LAWS OF MOTION

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THE LAWS OF MOTION

AN ELEMENTARY TREATISE

ON

DYNAMICS

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PREFACE

1. THE object of this treatise is to put the subject of Dynamics on a thoroughly sound basis, avoiding unsatisfactory illustrations and definitions which do nothing towards defining, and to endeavour to give the student such an accurate idea of the subject that he may be able *e.g.* to give explanations and illustrations of the laws without just merely copying these from the book.

2. As an example of a definition which does nothing towards defining:

Mass is sometimes defined as *quantity of matter;* but this (though I have found it help in some cases) conveys very little information.

Matter wants defining. It is defined as "that which can be perceived by the senses," or as "that which has the properties of impenetrability and extension," or again as "that which can be acted upon by, or can exert, force"; but under these definitions we might easily think that a cubic foot of elm contains the same "quantity of matter" as does a cubic foot of

oak; whereas, as a fact, the latter contains half the quantity again of "matter" as does the former.

3. Again, examinees quoting from the text-books constantly give the two following answers:

(i) Velocity = Rate of Motion;
(ii) Momentum = Quantity of Motion.

Now Velocity has no reference to Mass, whereas Momentum has; therefore "Rate of Motion" has no reference to Mass, whereas "Quantity of Motion" has. How can this be?

The fact is that in (ii) "Motion" is a technical word, and really is equivalent to Momentum; for when we speak of the Momentum of a body, if we mean the quantity of anything, we surely mean the quantity of momentum.

In (i) "Motion" is used in its ordinary sense.

I may say also that in the vague way in which "Motion" is used, there seems no reason why Kinetic Energy should not be called "Quantity of Motion."

Cases may easily be imagined in which there is Motion without Momentum; *e.g.* in the movement of a shadow.

This is naturally very puzzling to learners; and I have avoided altogether the use of the word "Motion" in any but its ordinary vulgar meaning.

4. On similar principles I have avoided any attempt at defining Force. [See Section V., paragraph 80.]

5. As an example of an unsatisfactory illustration, an illustration of the 2nd law of motion, which is given in text-books and constantly quoted by examinees, is :

A mass dropped from the top of the mast of a ship will fall at the foot of the mast whether the ship be at rest or in motion.

This does not illustrate the main part of the law, and to this fact the student's attention ought to be drawn ; and, as the mass does not fall in a straight line, it is not a very satisfactory illustration even of the second part.

6. The three[1] laws of motion are usually enunciated in a very imperfect manner.

7. The 1st law is given as: *There can be no*

[1] I have retained the *three* laws, merely giving them greater definiteness. But the 1st law is merely a particular case of the 2nd ; and it is to be hoped that eventually the present 1st law will be dropped, and the only laws set out be

The Law of Force (or Momentum) ;
and The Law of Work (or Energy).

change of motion without external force; or in equivalent words.

But if two balls collide there is considerable change of *motion*, though not of *momentum;* and "momentum" must be substituted, in the law, for "motion"; doing away with the seeming obviousness of the law, but adding greatly to its strength and character.

8. The usual omission of all reference to *Time* in the enunciation of the 2nd law makes that law positively untrue. A small force may produce much greater momentum than a larger force if it only have sufficient time.

9. Again, the 3rd law of motion is usually given as: *Action and Reaction are equal and opposite;* but what "Action" and "Reaction" are is left to the imagination of the student.

The following illustration is given: *If any one presses a table with his finger, his finger is pressed with the same force in the opposite direction by the table.*

This can hardly be an illustration of a law of *motion.*

The fact is that, if by "Action" and "Reaction" are meant Force and Resistance, the 3rd law is but

an easy deduction from the 2nd;[1] while, if D'Alembert's principle is really to be ultimately deduced from the law, it is better to at once enunciate it in proper form, and not in the usual indefinite and undefined terms.

10. I have endeavoured to keep the ideas of *Mass* and *Weight* distinct. There is great confusion in many books on this point, which it is important to clear away if only for the benefit of students who will apply dynamics to Astronomy.

For example, it is stated sometimes that "a definite mass has a definite weight."

But this is only because the mass is on the *surface* of the earth, and is not, *e.g.*, the earth itself, or at the centre of the earth. We might as well note as one of the characteristics of a mass that it has a definite amount of moisture on its surface without mentioning the fact that it is out in the rain.

11. I have made an attempt to abbreviate certain

[1] For suppose two systems to be connected by a rod; if this rod has mass there must be force to make it move in the direction of its length.

But if we may assume the rod to have no mass, then, by the second law, there can be no force in the direction of the rod's length; and therefore the force at each end must be the same.

If, then, we are considering the "Action and Reaction" at any point, we may isolate a very small portion of the mass, assume it to be massless, and "Action and Reaction are equal and opposite."

common groups of words. Thus "fas" is used to represent Foot A Second, etc. etc.

These words should be looked upon simply as abbreviations (perhaps in some cases as aids to the memory); I have no desire to add new words to the language.

12. As a small matter, it may be noticed that the examples are divided into groups; the first example in each group corresponding to "worked example" **A**, the 2nd to **B**; and so on.

13. I have to thank Mr. H. B. Goodwin, R.N., of the Royal Naval College, for much valuable criticism and many suggestions.

W. H. LAVERTY.

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SECTION I.

SIMPLE UNITS.

Time, Length, Mass.

1. THREE elementary conceptions of which it is impossible to give here any useful definitions are *Time, Length,* and *Mass.*

Time and *Length* are easily comprehended.

As to *Mass,* the following considerations help us to form some idea of it.

2. *Two masses* are equal which, at any particular spot of the earth, are, in ordinary parlance, "of equal weight"; *i.e.* would, if placed one in each scale of a balance, keep the balance level; or would, if weighed on a spring-balance, depress the pointer to the same mark.

3. Yet mass is very different from weight.

A mass is a constant quantity; but a cannon-ball which in London weighed (on a spring-balance) 32 lbs. 3 oz., would at the North Pole weigh 32 lbs. 4 oz., would at the Equator weigh 32 lbs. 1·4 oz., and would at the Centre of the Earth

weigh nothing. At all these places its mass would be the same.

So that weight is a variable quality of a body; but mass is a quality which never varies.

4. The great difficulty in conceiving of *mass* is to separate the idea from what we ordinarily call "weight."

When we "weigh" two bodies in *scales* we may be said to be

comparing their masses;

or we may be said to be

comparing their local weights.

If their "weights" are once shown equal by *a pair of scales,* they would be always so equal, wherever the earth exerted attraction.

But if "weighed" on a *spring-balance,* though their "weights" would be equal to one another at any place; yet the mark to which the pointer of the spring-balance would go would vary from place to place; *e.g.* at the centre of the earth they would each appear to have no "weight" at all.

And so their absolute "weights" would vary, but not their relative "weights."

5. When we "weigh" a "pound of sugar," we are simply finding that mass of sugar which is equal to a certain legal mass of iron; but, in doing so, we take advantage of the fact that, at any given place, the weight of each is the same.

6. We may say, then, that:

1. *Mass cannot easily be defined*; but the following are considerations which will help to fix the idea of it in the mind; viz.:
2. The *mass* of a body is *invariable*;
3. At any particular place two bodies of *equal mass* are also of *equal weight*.

7. The importance of clearly distinguishing between mass and weight may be seen from the following examples.

The weight of a mass (as we shall see in a subsequent section) is really the attraction of the mass by the earth.

Now, if a railway train is being pulled on a level road, the weight of the train is entirely supported by the road; and (if we neglect friction) will have nothing to do with the power of the engine required. But the mass of the train is an all-important consideration. The greater the mass the greater the power of the engine required to bring the train to a certain velocity. [Of course, if the train begins to go down-hill, the attraction of the earth, *i.e.* the weight of the train, will then help the engine.]

A man who can easily move a boat can make little impression on a large vessel, and this mainly on account of its mass. The resistance of the water has, of course, something to do with it, but the weight has no effect one way or the other.

8. We require some *standard* by which to measure *Time, Length,* and *Mass.*

9. The following statement is, of course, perfectly intelligible:

A weight of 3 *drams of powder will cause the bullet to travel with a velocity of* 1000 *yards in* $\frac{1}{20}$ *minute.*

unit & standard not the same

In this the standard (or *unit*, as it is usually called) of *Time* is *One Minute.* The unit of *Length* is *One Yard.* The unit of *Mass* is *One Dram.*

It is clear that Mass and not Weight is what is meant. For we clearly desire a certain "quantity" of powder, *i.e.* a certain mass of it; and weight being a variable thing, we might if we trusted to it obtain more or less than we desire.

10. The "British" unit of *Time* is the *Second* (the mean Solar Second).

This is the 3600th part of an hour; an hour being the 24th part of a "mean day."

This "mean day" is the time given by two complete revolutions of the hour-hand of an ordinary clock.

11. Noon being the epoch at which the Sun is due south, a Solar Day (called also an Apparent Day) is the time between noon and noon.

The length of this Solar Day is constantly varying,[1] but its average or *mean* length gives the 24 hours of the clock.

[1] This, of course, causes a constantly varying difference between Sun Time and Clock Time. In some almanacks this difference is given for every few days, and must be noted if we wish to tell the time by a Sun-dial.

12. The "*C.G.S.*"[1] unit of *Time* is also the *Second.*

13. The British unit of *Length* is the *Foot.*

The *Foot* is one-third of the "Imperial Standard Yard"; which is given by the distance between two fixed points marks on a certain metal rod kept in London, when the rod is at a certain temperature.

14. The *C.G.S.* unit of *Length* is the *Centimetre.*

15. The *Centimetre* is the one-hundredth part of the *Metre,* a length preserved in Paris.

The *Metre* is, theoretically, the ten-millionth part of the length of a meridian of the earth measured from the Pole to the Equator.

It is approximately equal to the length of the seconds-pendulum.

16. The Metre is between 3 and 4 inches longer than the Imperial Standard Yard. More exactly,

the metre = 1·093633 yards;
the centimetre = ·03281 feet;
the foot = 30·4794 centimetres;
the yard = ·914383 metres.

17. The British unit of *Mass* is the *Pound.*

The *Pound* is a certain piece of Platinum kept in London.

It is ordered that, if in the latitude of London this piece of platinum be put on a spring-balance, the mark to which

[1] The Centimetre-Gramme-Second System, as recommended by the British Association.

the pointer goes shall indicate that standard "weight" of "One Pound Avoirdupois" by which goods are to be bought and sold.

This mass is therefore called "a Mass of one Pound."

18. It is necessary to insist that this term, "a mass of one pound" is simply a convenient mode of expression if by "one pound" is meant a weight. The piece of platinum would remain "a mass of one pound" even if taken to the centre of the earth (where its weight would be nothing).

19. The *C.G.S.* unit of *Mass* is the *Gramme.*

The *Gramme* is the 1000th part of a Standard Platinum Kilogramme preserved in Paris.

1 gramme = 15·432 grains;
1 grain = ·06479895 grammes;
1 pound = 453·5926 grammes.

20. The subdivisions and multiples of the Second, Foot, and Pound (which is the pound avoirdupois) require no explanation.

21. In *C.G.S.* units:

Seconds, Minutes, Hours, etc., remain the same as in British Units.

In the case of measures of Mass:

1000	grammes	is called a	Kilogramme;
100	,,	,,	Hectogramme;
10	,,	,,	Decagramme;
1	,,	,,	Gramme;
1-10th	,,	,,	Decigramme;
1-100th	,,	,,	Centigramme;
1-1000th	,,	,,	Milligramme.

A similar system is adopted in the case of measures of Length; and the student will therefore notice that the unit of length, the Centimetre, is the 1-100th part of the Metre; and that a Kilometre is 1000 Metres, and therefore 100,000 Centimetres; and so on.

WORKED EXAMPLES.

A. How many metres are there in a mile, if there is ·305 of a metre in a foot? not suff. accur.

$$1 : 1760 \times 3 :: \cdot 305 : \text{required no.};$$

$\therefore$ there are 1610·4 metres in a mile.

1609.3

B. If a lb. be 453·6 grammes; how many kilogrammes are there in a ton?

$$1 : 2240 :: \frac{453\cdot 6}{1000} : \text{required no.};$$

$\therefore$ there are 1016·0~~64~~ kilogrammes in a ton.

C. A train goes 1 mile in one minute; how many centimetres does it travel in one second (1000 yards being taken as equivalent to 915 metres)?

	It goes	1760	yards	in 1 minute;
$\therefore$	,,	$\frac{1760}{60}$	,,	,, 1 second;
$\therefore$	,,	$\frac{1760}{60} \cdot \frac{915}{1000}$	metres	,, 1 ,, ;
$\therefore$	,,	$\frac{1760}{60} \cdot \frac{915}{1000} \cdot$	100 centimetres	,, 1 ,, ;
$\therefore$	,,	2684	centimetres	,, 1 ,, .

uniformly! i.e. is proportional to

D. Under certain circumstances of motion the mass moveable varies with the time; if, under such circumstances, 1 kilogramme can be moved in 1 second, how many minutes will it take to move 33 cwt.? [1 gramme = 15·4 grains.]

1 kilogramme is moved in 1″;

∴ 15400 grains are „ 1″;

∴ $\frac{11}{5}$ pounds „ „ 1″;

∴ 33 cwt. „ „ $33 \times 112 \times \frac{5}{11}$ seconds;

i.e. in 1680″;

i.e. in 28′.

E. If a uniform bar of iron 1 foot long masses[1] 1 pound; what length in centimetres of the same bar will mass 151 grammes (1000 yards being taken as equivalent to 915 metres; and 1 pound as ·453 kilogramme)?

1 foot masses 1 pound;

∴ 1000 yards mass 3000 pounds;

∴ 915 metres „ 1359 kilogrammes;

∴ 91500 centimetres „ 1359000 grammes;

∴ $\frac{91500}{1359000} \times 151$ „ „ 151 „ ;

∴ req. no. of centimetres $= \frac{915 \times 151}{13590} = 10\tfrac{1}{6}$.

EXAMPLES.—I.

1. How many centimetres are there in 1 foot, if 1 yard = ·91438 of a metre?

2. How many grains are there in a kilogramme, if a gramme contains ·0022 lb.?

[1] *i.e.* has a mass of.

3. If a train goes 1 mile in 1 minute, how many feet does it go in one second?

4. If the greater the mass to be impressed, the greater the time required in direct proportion; and if a certain impression can be made on 63 kilogrammes in 1′ 40″; on how many lbs. can the same impression be made in 1′ 12″? [1 pound $=453\frac{3}{5}$ grammes.]

5. If a uniform bar of iron, 10 centimetres long, masses 1 kilogramme; find in pounds the mass of a similar bar of the same section 16·57 feet in length. [Take 1 kilometre as equivalent to 1093·62 yards, and 1 gramme to 15·4 grains.]

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fraction

6. What decimal of a foot is a centimetre? [1 metre = 1·0936 yards.]

7. If a kilogramme be taken as equivalent to $2\frac{1}{5}$ lbs.; how many grammes does this give to 11 ounces?

8. In falling bodies the number of units of length through which the body falls varies as the square of the number of units of time. If a body fall through 16 feet in 1″; in what time will it fall 488 metres? [1 foot $=30\frac{1}{2}$ centimetres.]

9. If the momentum required to cover a certain length is directly proportional to the mass, and inversely proportional to the time to be occupied; and if a certain momentum be required to cause 1 ton to cover a given length in 1 hour; in what time would the same length be covered by 28 kilogrammes endowed with the same momentum? [1 gramme $=\cdot0022$ lbs.] what is m.?

10. If, of two uniform bars of equal section of iron and copper, a yard of the iron bar masses 3 lbs., and a metre of the copper

bar masses $1\frac{1}{2}$ kilogrammes; which has the more mass per *C.G.S.* unit of length, and by what decimal of a gramme? [1 metre=3·28 feet; and 1 lb.=453·6 grammes.]

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11. How many inches are there in a centimetre? [1 metre= 1·0936 yards.]

12. A pound being 453·6 grammes; how many ounces are there in 81 kilogrammes?

13. A man's speed being reckoned as directly proportional to the number of units of length passed over, and inversely proportional to the number of units of time taken; if a man, who walks with a certain speed, covers $7\frac{1}{2}$ kilometres in 1 hr. 4 min. 20 sec., in what time does he walk 102 yards? [1 metre=3·281 feet.]

14. If the energy required to cover a certain length is directly proportional to the number of units of mass, and inversely proportional to the square of the number of units of time to be occupied; and if a certain energy be required to cause 100 kilogrammes to pass over a given length in one minute; how many pounds endowed with the same energy would pass over the same length in 18 seconds? [1 gramme=·0022 lb.]

15. If a yard of wire masses 33 ounces, how many kilogrammes are there in 61 metres? [1 gramme=·0022 lb.; 1000 yards=915 metres.]

SECTION II.

DERIVED UNITS.

Area, Volume, Density.

22. THESE fundamental units of *Time, Length,* and *Mass* being determined, other kinds of quantities will have their units fixed.

23. Thus the Scientific British unit of Length being the Foot, the unit of *Area* is the *Square Foot,* and the unit of *Volume* is the *Cubic Foot.*

24. So if the side of a square be l feet, the area of the square is l^2 square feet.[1]

If the edge of a cube be l inches, the volume of the cube is l^3 cubic inches.

If the sides of a rectangle be l and l' yards, the area of the rectangle is ll' square yards.

And so on.

25. Similarly for the *C.G.S.* units of area and volume; which are respectively a *square centimetre,* and a *cubic centimetre.*

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[1] See the note to paragraph **41**.

It is obvious that a metre being equal to 100 centimetres, a square metre is equal to 10,000 square centimetres.

And, again, that a decimetre being 10 centimetres, a cubic decimetre is 1000 cubic centimetres.

not always

26. A cubic decimetre is called a *Litre.*

A Litre contains about $1\frac{3}{4}$ pints; more exactly

1 Litre = ·035317 cub. ft. = ·2200967 gallons.

27. The same system is adopted as with the gramme and the metre; thus:

1 Kilolitre = 1000 Litres = about 4 Hogsheads;
1 Hectolitre = 100 ,, = ,, $26\frac{1}{2}$ gallons (wine measure);

and so on.

28. It may also be noticed that an area of 100 square metres is called an *Are*; and a *Hectare* contains 100 *Ares*; and so is equivalent to nearly $2\frac{1}{2}$ acres.

29. Again, the units of mass and of volume being fixed, that of *Density* will also be fixed.

Density depends upon the quantity of volume in a given mass.[1] The less the volume in the given mass the greater the *Density.*

DEFINITION.—The *Density* of a substance varies inversely as the quantity of volume in a given mass of the substance.

imagine

[1] The student is recommended to put before himself, on the table, a mass of 1 lb. (say an ordinary iron "weight" of 1 lb.).

30. Now that amount of *Water* (distilled and at a certain temperature and pressure) which "weighs" a gramme, *i.e.* which has a mass of a gramme, has a volume of 1 cubic centimetre.

So in *C.G.S.* units there is a unit volume of water in a unit mass.

So in these units *Water* is of unit density.

31. There is about one-seventh of the volume in a given mass of *iron* that there is in the same mass of water;

i.e. if we had a cup which exactly held the 1 lb. iron mass, it would require to be filled 7 times with water to balance the 1 lb. in the scales. Much ado about nothing

So the density of iron is 7 times the density of water.

And ∴ in *C.G.S.* units the density of iron is 7 times the unit density.

32. Oak is "heavier" than Elm, and so has a greater mass.

By "weighing" a cubic foot of oak and a cubic foot of elm, we find that their masses are in the proportion of 31 to 20; and so

Density of Oak : Density of Elm : : 31 : 20.

33. In general, the *Unit of Density* is the density of a substance in which there is unit of volume in unit of mass.

So the *British Unit of Density* is the density of that uniform substance of which a cubic foot masses 1 pound.

34. In like manner the *C.G.S. Unit of Density* is the density of that substance of which a cubic centimetre masses a gramme.

This substance, as we have seen, is water.

35. With British Units there is, of course, very much less than a unit volume in a unit mass of water.

In fact there is approximately $62\frac{1}{2}$ times the lb. mass in the cubic foot.[1]

The density of water (with British Units) is therefore not the unit density.

36. Nevertheless it is a convenient standard, and the density of solids and liquids is usually measured with reference to it.

There is no actual substance which has the unit density; the density of such a substance as Cork being 15 times the unit density.

37. Thus Platinum having a mass 22 times that of water is said to have a *Specific Density* of 22.

Its density in British units is $22 \times 62\frac{1}{2}$ times the unit density.

38. The density of Cork is 15 times the unit density; its *Specific Density* is $\therefore$ $15 \div 62\frac{1}{2}$, or rather less than $\frac{1}{4}$.

---

[1] So it is commonly said that a cubic foot of water "weighs" $62\frac{1}{2}$ pounds, or 1000 ounces.

More exactly, there are 27·7274 cubic inches of distilled water at 60° Fahrenheit in a 1 lb. mass.

of water

39. Definition.—The *Specific Density* of any substance is the ratio of the density of the substance to that of water.

[With *C.G.S.* units, but not with British units, the number which represents the specific density also gives the number of units of density.]

40. This ratio is often called the *Specific Gravity* of the body; but (to use the same illustration as before) the ratio would still be an existing fact even at the centre of the earth, where there is no "gravity."

41. If there be c cubic centimetres in a mass of m grammes, and if its density be d times that of water, then

$$m = dc.$$[1]

[d is here, of course, both the Specific Density and the number of Units of Density.]

42. With British units this equation is not true if d be the Specific Density.

If there be c cubic feet in a mass of m lbs., and if ρ be its Specific Density, then approximately

$$m = 62\tfrac{1}{2}\rho c.$$

43. To recapitulate:—We have three fundamental conceptions, *Time, Length, Mass,* which we cannot usefully

[1] With reference to this and other like equations, and to the expressions l^2, l^3, ll' (used in Paragraph **24**), and others of similar character, it cannot be too strongly impressed upon the student that m, ρ, d, c, l, l^2, l^3, ll', and other like symbols *are only numbers.*

define, except that we may say that the *Mass* of any substance is very much what, in common talk, is usually called the "Weight" of it.

44. We have settled on *Units* of Time, Length, and Mass; the *British* being a *Second*, a *Foot*, and that which in London "weighs" a *Pound* Avoirdupois; and the *C.G.S.* being a *Second*, a *Centimetre*, and that which in Paris "weighs" a *Gramme.*

45. The *Units* of *Space* and *Volume* naturally follow on that of Length.

46. As to *Density*: we have seen that *Water* is a substance which with *C.G.S.* units is of unit density. With *British* units its density is about $62\frac{1}{2}$ times the unit density; but it makes a convenient standard, so that the densities of solids and liquids are usually compared with its density.

The number which, for any substance, expresses this comparison is called the Specific Density of the substance.

WORKED EXAMPLES.

A. A litre being a cubic decimetre, and a gallon 4540 cubic centimetres; how many pints are there in $567\frac{1}{2}$ kilolitres?

1 decimetre $=10$ centimetres;

$\therefore$ 1 cubic decimetre $=10^3$ cubic centimetres;

also 1 kilolitre $=10^3$ litres;

$\therefore 567\frac{1}{2}$ kilolitres $=567\frac{1}{2}\times 10^3$ cubic decimeters;

$$=\frac{567\frac{1}{2}\times 10^6}{4540} \text{ gallons};$$

$$=\frac{567\frac{1}{2}\times 10^6\times 8}{4540} \text{ pints};$$

$=$ one million pints.

B. If a kilogramme is $2\frac{1}{5}$ lbs., and if there are 20 oz. in a pint, and 22 gallons in a hectolitre; how many grammes are there in a centilitre?

1 hectolitre $=100$ litres $=10,000$ centilitres;

now 1000 grammes $=1$ kilogramme;

$$=\frac{11}{5}\times\frac{16}{20} \text{ pints};$$

$$=\frac{11}{5}\times\frac{16}{20}\times\frac{1}{8\times 22} \text{ hectolitres};$$

$$=\frac{11}{5}\times\frac{16}{20}\times\frac{10000}{8\times 22} \text{ centilitres};$$

$=100$ centilitres;

$\therefore$ there are 10 grammes in 1 centilitre.

C. Water (of which a cubic foot masses 1000 oz.) being the substance of unit density, and the unit of length being 6 inches; what is the unit of mass?

1 cubic foot of unit substance masses 1000 oz.;

$\therefore$ 1728 cub. ins. „ „ 1000 oz.;

$\therefore$ 6^3 cub. ins. „ „ $\frac{1000}{1728}\times 216$ oz.;

$\therefore$ unit volume masses $\frac{1000}{8}$ oz. $=125$ oz.

D. A cubic foot of water and a cubic foot of brass massing respectively 1000 oz. and 500 lbs.; find in British Units the density of the brass, and its specific density?

The specific density is, of course, $\frac{500 \text{ lbs.}}{1000 \text{ oz.}} = 8.$

For the density $m = dc$;

where $\begin{Bmatrix} m \\ c \end{Bmatrix}$ must be *numbers* of $\begin{Bmatrix} \text{lbs.} \\ \text{cub. ft.} \end{Bmatrix}$; and d is a number;

$$\therefore 500 = d.1;$$

$$\therefore \text{density} = 500.$$

E. The unit mass being a pound and the unit density $\frac{2}{3375}$ of that of water; find the unit of length. [A cubic foot of water masses 1000 oz.]

If l feet be the unit of length;

$$\frac{1}{l^3} \text{ units mass } 62\tfrac{1}{2} \text{ lbs.};$$

$\therefore$ the equation, $m = dc$, is, for water,

$$62\tfrac{1}{2} = \frac{3375}{2} \frac{1}{l^3};$$

$$\therefore l = \sqrt[3]{\frac{3375}{125}} = 3;$$

and the unit is 3 feet, or 1 yard.

F. A nugget of gold and quartz having a mass of 12·58 ounces avoirdupois and a specific density 7·4; and the specific densities of gold and quartz being respectively 19·4 and 2·4; find the mass of gold in the nugget.

If G and Q be the masses of gold and quartz in ounces;

$\frac{G}{16 \times 62\frac{1}{2} \times 19{\cdot}4}$ and $\frac{Q}{16 \times 62\frac{1}{2} \times 2{\cdot}4}$ will be the volumes of gold and quartz in cubic feet;

also the mass and volume in the nugget are respectively 12·58 oz.;

$$\text{and } \frac{12{\cdot}58}{16 \times 62\frac{1}{2} \times 7{\cdot}4} \text{ cubic feet;}$$

$$\therefore \quad G+Q = 12{\cdot}58;$$

$$\text{and } \frac{G}{19{\cdot}4} + \frac{Q}{2{\cdot}4} = \frac{12{\cdot}58}{7{\cdot}4};$$

whence we easily obtain $G = 9{\cdot}7$.

EXAMPLES.—II.

1. A decimetre being the 10th part of a metre; and a litre being a cubic decimetre; find, to 4 decimal places, how many cubic feet there are in a litre. [Take 1 centimetre = ·3936 inches.]

2. A bushel of oats masses 40 pounds; what part of a kilogramme is that to the litre? [Take 1 kilogramme = 2·2 lbs.; 1 litre = ·22 gallon.]

3. Taking 1000 yards as equal to 915 metres; find the number of kilogrammes in a cubic foot of water.

4. Taking a pound as $453\frac{1}{2}$ grammes; and a foot as $30\frac{1}{2}$ centimetres; find approximately the relation between the British and *C.G.S.* units of density.

5. A cubic foot of water massing 1000 oz.; if the density of water be the unit density, and $\frac{1}{2}$ lb. the unit of mass; find the unit of length.

6. 35 lbs., 45 lbs., and 39 lbs. 6 oz. of three fluids whose specific densities are respectively ·7, ·8, and ·9, are mixed together. Find the specific density of the mixture.

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7. There being 34·66 cubic inches in a pint; and 2·54 centimetres in an inch; how many cubic centimetres are there in a gallon?

8. If a litre of oil masses 33 ounces; how many grammes are there in a gallon? [100 litres = 22 gallons; 1 lb. = 453·6 grammes.]

9. What is the mass in cwts. of a cubic metre of a substance whose specific density is ·56? [1 kilogramme = 2·2 lbs.]

10. A cubic foot of water and a cubic yard of coal mass respectively 1000 oz. and 18 cwt. 3 qrs. 9 lbs. 6 oz.; what is the specific density of the coal?

11. A mass of 125 ounces of ash (specific density $\frac{4}{5}$) floats in water (of which a cubic foot masses 1000 ounces). It being given that if a body floats in any fluid the mass of the body is equal to the mass of the fluid displaced by it; how many cubic inches of the ash will be above the surface of the water?

12. Two metals are such that 6 cubic inches of the one, 10 cubic inches of the other, and 8 cubic inches of a certain combination of them, all mass the same. How many cubic inches of the first metal are there in the combination?

---

13. Two cylinders of equal length have, the one a radius of 1 inch, and the other a radius of $2\frac{1}{2}$ centimetres. If the volume

of the latter is a litre, what English measure is equal to the volume of the former? [1 foot = 30·48 centimetres; and 4 litres = 7 pints.]

14. If a litre of water masses a kilogramme; how many ounces is this to the pint? [1 kilogramme = 2·2 lbs.; 1 litre = ·22 gallons.]

15. Taking the earth as a sphere, of radius 4000 miles and specific density $5\frac{8}{11}$; and supposing it to contain 8 billion units of mass; find the units of mass and length; the unit density being 12 times the British unit of density. [$\pi = \dfrac{22}{7}$; and a cubic foot of water masses 1000 ounces.]

16. A cubic foot of water massing 1000 ounces; compare the densities of two substances of which a cubic centimetre of one masses 6 grammes, and a cubic inch of the other masses $3\frac{1}{2}$ ounces.

17. It being given that if a body floats in any fluid the mass of the body is equal to the mass of the fluid displaced by it; determine the volume and specific density of a piece of iron, massing 300 grammes, which floats in mercury, of specific density $13\frac{1}{2}$, with $\frac{5}{9}$ths of its volume immersed.

18. A coin, whose mass and specific density are respectively ·629 oz. and 15·725, is composed of gold (specific density $17\frac{1}{2}$) and silver; the relative volumes being 3 of gold to 1 of silver. Find the mass of each metal.

SECTION III.

DERIVED UNITS.

Velocity, Acceleration.

47. The derived units in Section II. have only involved Length and Mass.

The units in this section will both involve *Time*; and will not involve Mass.

48. Velocity is a simple combination of the elementary units; and requires no definition.

cf. accel. art. 55

Velocity is measured by the *Length* described in a given *Time*.

49. Thus, a railway train is said to travel with a Velocity of 30 miles an hour.

Light comes from the Sun with a Velocity of 12 million miles a minute.

50. The British *Unit of Velocity* is, of course, the velocity of 1 Foot in 1 Second. This we may call a *Fas.*[1]

The *C.G.S.* Unit of Velocity is one of 1 centimetre in 1 second. This we may call a *Cas.*[2]

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[1] a **F**oot **A** **S**econd.

[2] a **C**entimetre **A** **S**econd.

51. As there are about $30\frac{1}{2}$* centimetres in a foot, the British Unit of Velocity contains about $30\frac{1}{2}$ of the *C.G.S.* units; or 1 fas $= 30\frac{1}{2}$ cas.

52. If v be the number of fas when a length of l feet is described in t seconds, then

$$\text{since } 1 \text{ fas} = \text{a velocity of } 1 \text{ foot per } 1'';$$

1" means 1s!

$$\therefore\ l \text{ fas} = \quad \text{,,} \quad l \text{ feet per } 1'';$$

$$\therefore\ \frac{l}{t} \text{ fas} = \quad \text{,,} \quad l \text{ feet per } t'';$$

[because the longer the time taken to cover l feet the less the velocity;]

$$\therefore\ v = \frac{l}{t};$$

$$\therefore\ l = vt. \qquad \text{(1)}$$

53. The same equation will hold, of course, for *C.G.S.* units.

54. This equation applies only to a *constant* velocity.

Velocity may vary from instant to instant. Thus a falling mass is constantly and equably increasing its velocity. If such a mass fall from rest, and in T seconds fall through a height of H feet; and if U fas be its velocity at the end of the T seconds; then, since the velocity has increased equably,

$$\text{the average velocity} = \frac{U}{2} \text{ fas};$$

and $\therefore$ the equation above becomes

$$\frac{1}{2}U = \frac{H}{T}.$$

And so in other cases.

* More correctly 30·4794.

55. Acceleration is another combination of the same elementary units.

DEFINITION.—*Acceleration* is the change in any *Time* of any *Velocity*.

Acceleration is measured by the *Velocity* gained or lost in a given time.

We shall deal with *Constant Accelerations* only.

56. Thus, the earth attracts all bodies near its surface in such a way as, every second, to increase the Velocity of the body by g Fas; where g is a number ($>32{\cdot}088$ and $<32{\cdot}255$), depending on the latitude of the place.

E.G. at Edinburgh g is very nearly equal to 32·2.

57. The British *Unit of Acceleration* will be that of a body which in 1″ increases its velocity by a Fas.

This we may call a *Sfas.*[1]

With British units, therefore, the attraction of the earth produces an acceleration of a little more than 32 sfas.

58. Similarly for the *C.G.S. Unit of Acceleration*; which we may call a *Scas.*[2]

59. To find the value of g with *C.G.S.* units, at a place where its value with British units is 32·1.

---

[1] The increase in a Second by a **FAS**.

[2] The increase in a Second by a **CAS**.

The earth's attraction produces

an additional velocity in 1″ of 32·1 feet per . . . . 1″;

i.e. " " 1″ " $\frac{32\cdot1}{3}$ yards per . . . . 1″;

i.e. " " 1″ " $\frac{32\cdot1}{3}\cdot\frac{915}{1000}$ metres per . . 1″;

[a yard : a metre :: ·915 : 1]

i.e. an additional velocity in 1″ of $\frac{(32\cdot1)\times305}{10}$ centimetres per 1″;

i.e. " " 1″ " 979·05 centimetres per . 1″;

and the value of g is accordingly 979·05.

60. As a fact, in *C.G.S.* units,

$$g > 978\cdot1 \text{ and } < 983\cdot11.$$

61. If a be the number of units of an acceleration which will in t'' increase the velocity by v fas; then since

unit acceleration = the addition in 1″ of 1 fas;

∴ v units of acceleration = " " 1″ " v fas;

∴ $\frac{v}{t}$ " " = " " t'' " v fas;

[because the longer the time in which the v fas are added, the less the required number of units of acceleration];

but a units of acceleration = the addition in t'' of v fas;

$$\therefore a = \frac{v}{t};$$

$$\text{or } v = at \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

62. If x be the total number of feet passed over in the t seconds; then, noticing that the velocity at the beginning of

the time was 0, and at the end of the time v fas; and that, therefore, the space would have been equally well described by a uniform movement of $\frac{1}{2}v$ fas;

$$\therefore\ x=\frac{v}{2}t;$$

$$\therefore\ x=\frac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We may also, by eliminating between equations (2) and (3), obtain the equation

$$\frac{1}{2}v^2=ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

63. So, generally, if a be the number of units of an acceleration under which a body in t units of time moves through a distance of x units of length,

$$x=\frac{1}{2}at^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, if v units of velocity be the total increase of velocity,

then, in terms of the *time*,

$$v=at; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, in terms of the *distance*,

$$\frac{1}{2}v^2=ax \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

64. It is again earnestly pressed upon the student to remember that all these symbols are only numbers.

65. The forms of equations (2) and (4) should be carefully noted, for the better understanding of the Laws of Motion presently to be given.

WORKED EXAMPLES.

A. If the units of area and time be 10 acres and 10 seconds; what is the unit of velocity expressed in miles per hour?

Unit velocity is $\sqrt{48400}$ yards per 10 seconds;

$\therefore$,, ,, $\frac{220}{1760} \times 6 \times 60$ miles per 1 hour;

$\therefore$,, ,, 45 miles per hour.

B. At a place where the acceleration of gravity is 980 scas, find the value of g when the units of time and velocity are 10 minutes and $1\frac{2}{3}$ cas.

A vel. of $\frac{5}{3}$ cas = a vel. of $\frac{5}{3} \times 60$ centimetres per 1′;

= a vel. of 1000 ,, ,, 10′;

Now the accel. = an addition in 1″ of 980 centimetres per 1″;

$\therefore$,, = ,, ,, 10′ of $980 \times (600)^2$ centimetres per 10′;

$\therefore$,, = ,, ,, 10′ of $\frac{980 \times (600)^2}{1000}$(1000 cms.) per 10′;

$\therefore$ the value of g is 352800.

C. At a place where the acceleration of gravity is 32·12 sfas; what is taken as the unit of time when, with a mile as unit of length, g is 87·6?

The earth causes an accel. in 1″ of 32·12 feet per 1″;

$\therefore$,, ,, 1″ ,, $\frac{32\cdot12}{3 \times 1760}$ miles ,, 1″;

$\therefore$,, ,, t'' ,, $\frac{32\cdot12}{5280}t^2$,, ,, t'';

and we must have $\frac{32\cdot12}{5280}t^2 = 87\cdot6$;

$\therefore$ $t^2 = 14400$;

$\therefore$ $t'' = 120''$;

$= 2$ minutes.

D. Given that a body on a certain smooth inclined plane has an acceleration of 109 scas; find what its velocity will be after it has passed over 872 centimetres; and the time of passage.

For the first part of the question we may use the equation

$$\frac{1}{2}v^2 = ax.$$

$$\text{So } \frac{1}{2}v^2 = 109 \times 872;$$

whence velocity $= 436$ cas;

cas, of course; since the acceleration and length are expressed in scas and centimetres.

For the time

$$t'' = \frac{v}{a} = \frac{436}{109} = 4 \text{ seconds};$$

or, again,

$$t'' = \sqrt{\frac{2x}{a}} = \sqrt{\frac{1744}{109}} = 4 \text{ seconds}.$$

E. If a particle which has fallen freely from rest passes over 161 feet in a certain second, at a place where, with British units, $g = 32{\cdot}2$; how long had it been falling before the beginning of the second?

This is an example of the equation $x = \frac{1}{2}at^2$. If t'' be the time before the beginning of the second; then, evidently,

$$161 = \frac{1}{2}(32{\cdot}2) \times \{(t+1)^2 - t^2\};$$

$$\therefore \quad 2t + 1 = 10;$$

and the required time is $4\frac{1}{2}$ seconds.

EXAMPLES.—III.

1. If when the units of length and time are a mile and an hour, a particle is moving with 1200 units of velocity; with what number of fas is it moving?

2. The acceleration of gravity being 32 sfas, express it in scas. [1000 yards = 915 metres.]

3. If unit acceleration be 32 sfas, and unit velocity be a mile a minute; find the units of length and time.

4. If the units of acceleration and velocity be 32 sfas and 1 mile in 10 minutes; find the number of feet passed over in a unit of time from rest.

5. Find the feet through which a particle will fall freely in 3 seconds at a place where the acceleration of gravity equals 32·12 sfas.

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6. If the earth described a circle of 91,000,000 miles' radius in 366 days; how many metres a second would that be? [1000 yards = 915 metres; $\pi = \frac{22}{7}$.]

7. The acceleration of gravity being 32 sfas; what number represents it when the units of length and time are 1 yard and 1 minute?

8. If 10 acres be the unit of area, and the acceleration of gravity ($g = 32$, British Units) the unit acceleration; find the unit of time.

9. If a particle move from rest with uniform acceleration through 372·1 centimetres, and acquire a vel. of 30 fas; what, in scas, is the acceleration? [1 foot = $30\frac{1}{2}$ centimetres.]

10. Find the centimetres passed over in the 5th second from rest, by a particle falling freely in a place where the acceleration of gravity is equal to 980 scas.

---

11. Two particles start from a vertex of an equilateral triangle, and move uniformly along the two sides with velocities of $65\frac{1}{2}$ fas and 1732 cas; how many metres apart (approximately) will the particles be at the end of 10 seconds? [1 foot $= 30\frac{1}{2}$ cms.]

12. The unit of acceleration being $\frac{2}{15}$ sfas; find the unit of time when the unit of velocity is a mile an hour.

13. At a place where the acceleration of gravity is 32·13 sfas, how far will a mass fall *in vacuo* in $3\frac{1}{3}$ seconds; and what is the velocity acquired?

14. A particle in passing over h feet has its velocity increased by h fas; show that the number of units of acceleration is the Arithmetical Mean between the numbers of units of velocity at the beginning and end of the h feet.

15. Show that the length described by a particle moving from rest during a given time under a uniform acceleration, is equal to half the length that would be described in the time with the final velocity.

---

16. Two trains of the same length passed through a station with the same velocity, one on a down and the other on an up line. The one took 5 seconds to pass a passenger in a train moving at $7\frac{1}{2}$ miles an hour, the other took but 3 seconds. At how many miles per hour were they moving?

17. If the unit of time be 1 minute, and the unit of length 1 mile; find the number of units of acceleration in 32 sfas.

18. Find the units of time and space when a particle moving from rest with F units of acceleration describes L feet in T_1 seconds; and at the end of T_2 seconds is moving with V units of velocity.

19. If, at a place where the acceleration of gravity is 980 scas, a particle in falling through 140 centimetres has its velocity increased by 140 cas; find its velocity at the beginning of the 140 centimetres, and the space the particle had then fallen from rest.

20. If the surface of the water in a well be 90 metres below the ground, and if the velocity of sound be 35,000 cas; what time will elapse, after the dropping of a stone, before the sound of the splash is heard; at a place where the acceleration of gravity is 980 scas?

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21. What is the velocity due to the rotation of the earth (i) in fas, (ii) in cas, of a man at the equator? [Take the earth's equatorial circumference as 24,900 miles; its rotation as once in 24 hours; and 1000 yards as equivalent to 915 metres.]

22. The attraction of the earth causing at a certain place an acceleration per day of 45,411,840 miles per day; how many sfas is this?

23. What are the units of time and length, if, under an acceleration equal to 20 times the unit acceleration, a particle moving from rest for 3″ describes $2\frac{1}{2}$ centimetres, and acquires the unit velocity?

24. If the units of velocity and acceleration be respectively l feet in t'', and that of a point which acquires $\frac{\lambda}{\tau}$ fas in τ''; find the number of feet that will be passed over in a unit of time from rest.

25. Find the units of length and time, when a particle moving under 10 units of acceleration, covers 264 feet in 1 second, and acquires the unit of velocity.

SECTION IV.

DERIVED UNITS.

Momentum, Kinetic Energy, Mass-Acceleration.

66. BEFORE we have a complete picture of the motion of a body we must introduce the consideration of its mass as well as that of its velocity.

67. We are familiar with the fact that if a cannon-ball and a football were moving side by side with the same velocity the former would be very much more difficult to stop than the latter; and this because its MASS is greater.

Again, of two cricket-balls the one which is moving with the less VELOCITY is the easier to stop.

So that the knowledge of both the mass and the velocity is necessary for a complete picture of the motion.

68. In order to see what combinations of mass and velocity it may be useful to take, consider the simple case of a constant acceleration.

As in the previous section, if a be the number of units of a constant acceleration which will in t'' increase the velocity

by v fas; and if x be the total number of feet passed over in the t''; then

$$v = at;$$

$$\frac{1}{2} v^2 = ax.$$

If, now, m be the number of lbs. of mass in the body which is being accelerated, we may write these equations,

$$mv = ma \times t;$$

$$\frac{1}{2} mv^2 = ma \times x.$$

69. We shall see in the succeeding sections that these two last equations are simple forms of the 2d and 3d Laws of Motion; and they serve to show us which combinations we may with advantage take.

These are represented by

i. mv;

ii. $\frac{1}{2} mv^2$;

iii. ma;

which respectively represent

i. Momentum;

ii. Kinetic Energy;

iii. Mass-Acceleration.

70. So that

No. of units of momentum = no. of units of mass-acceleration × no. of units of time;

No. of units of kin.-energy = no. of units of mass-acceleration × no. of units of length.

71. Whence, also, or from the equation

$$mx = \frac{1}{2} ma \times t^2,$$

we derive a relation between the numbers of units of mass, length, mass-acceleration, and time.

72. Definition.—The *Momentum* of a body is that quality the number of units of which is equal to the *product* of the number of units of *mass* by the number of units of *velocity.*

So if a body with m units of mass, move with v units of velocity; then if k represent the number of units of momentum with which it is endowed

$$k = mv.$$

73. The British *Unit of Momentum* is given by the moving of a mass of one Pound with a velocity of one Fas.

This we may call a *Fasp.*[1]

The *C.G.S. Unit of Momentum* is given by the moving of a mass of one Gramme with a velocity of one Cas.

This we may call a *Casgram.*[2]

74. Definition.—The *Kinetic Energy* of a body is that quality the number of units of which is equal to *half* the *product* of the number of units of *mass* by the *square* of the number of units of *velocity.*

So if a body with m units of mass, move with v units of velocity; then if η represent the number of units of kinetic energy with which it is endowed

$$\eta = \frac{1}{2} mv^2.$$

[1] A momentum of a **FAS** in a **P**ound.

[2] A m‹ nentum of a **CAS** in a **GRAM**me.

75. We have the British *Unit of Kinetic Energy* when a mass of one Pound moves with a velocity of one Fas; and the *C.G.S. Unit of Kinetic Energy* when one Gramme moves with a velocity of one Cas.

These we may respectively call a *Faspen* and a *Casgrammen*.[1]

76. Definition.—The *Mass-Acceleration* of a body is that quality the number of units of which is equal to the *product* of the number of units of *mass* by the number of units of *acceleration.*

So of a body of m units of mass moving with a units of acceleration, the number of units of mass-acceleration is

$$ma.$$

77. The Unit of Mass-Acceleration is given when unit mass moves with unit acceleration; or, as we may clearly say, is the acceleration of unit mass in unit time by unit velocity.

So the British *Unit of Mass-Acceleration* is the acceleration of one Pound in one Second by one Fas.

This we may call a *Sfasp.*[2]

The *C.G.S. Unit of Mass-Acceleration* is the acceleration of one Gramme in one Second by one Cas.

This we may call a *Scasgram.*[3]

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[1] Obviously from **FASP** and **EN**ergy; and **CASGRAM** and **EN**ergy.

[2] The acceleration every **S**econd by one **FAS** of one **P**ound.

[3] The acceleration every **S**econd by one **CAS** of one **GRAM**me.

WORKED EXAMPLES.

A. It being given that, when two inelastic balls collide, their velocities are equal after the collision, and the total momentum of the system remains the same; find the common velocity after collision of two inelastic balls of masses 6 and 4 kilogrammes, which were moving with velocities respectively of 250 and 150 cas.

If v be the common velocity required,

$$6000v+4000v=6000\times 250+4000\times 150;$$

whence velocity $=210$ cas.

B. Two inelastic balls of masses m and m' pounds are moving in the same direction; prove that the difference of the kinetic energies of the system before and after impact is equal to the kinetic energy of m moving with the velocity it loses, together with that of m' moving with the velocity it gains. [The momentum of the system remains the same; and after impact the velocities of m and m' are equal.]

If v and v' fas be the velocities before, and V fas the common velocity after impact, we have to show that

$$\left(\frac{1}{2}mv^2+\frac{1}{2}m'v'^2\right)-\left(\frac{1}{2}mV^2+\frac{1}{2}m'V^2\right)=\frac{1}{2}m\left(v-V\right)^2+\frac{1}{2}m'\left(V-v'\right)^2;$$

which is true if

$$0=mV^2+m'V^2-mvV-m'v'V;$$

$$i.e. \text{ if } \quad mV+m'V=mv+m'v';$$

which is true by the data.

C. How many scasgram are equivalent to 32 sfasp? [1000 yards $=915$ metres; 1 pound $=453$ grammes.]

32 sfasp	is the accel. of	1 lb. in 1″	by	32 ft. per 1″;
∴ 32	,, ,,	453 gr. ,, 1″	,,	976 cm. ,, 1″;
∴ 32	,, ,,	1 gr. ,, 1″	,,	976 × 453 cm. ,, 1″;

i.e. the number of scasgram required is

$$976 \times 453 = 442128.$$

D. At a place where the acceleration of gravity is 981 scas, find the units of time, length, mass, momentum, and kinetic energy; when a mass of 100 kilogrammes moving with a velocity of 36 cas has 20 units of momentum and 2 units of kinetic energy; and when the acceleration of gravity is 10·9 units of acceleration.

Note on Units.—In dealing with units, changing units, finding units, etc., some care will be necessary.

We must not say

$$1 \text{ faspen} = \frac{1}{2}(1 \text{ pound}) \times (1 \text{ fas})^2;$$

$$\text{or unit energy} = \frac{1}{2}(\text{unit mass}) \times (\text{unit vel.})^2;$$

such equations are absolute nonsense.

You cannot multiply a pound by a fas or divide a faspen by either, any more than you can multiply pounds sterling by square feet; (though, of course, we may use such expressions as well-understood abbreviations when we are familiar with the subject).

Now the unit will be required in terms of some well-known unit, as the British or *C.G.S.*

Suppose it to be required in terms of a British unit.

Form two equations, the one in terms of the given and required units, the other in terms of British units.

[These equations will, of course, be equations in numbers only.]

From these two equations the number required will be found.

Thus let 10 lbs. be the unit of mass, and 7 ft. the unit of velocity; and let it be required to find the unit of kinetic energy.

Let E faspen be this unit; then in a case in which there were

m units of mass;
v „ velocity;
η „ kinetic energy;

we should have, of course, as usual

$$\eta = \frac{1}{2}mv^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is an equation expressed with reference to the given units.

The same thing expressed with reference to British units is

$$E\eta = \frac{1}{2}\,10m \times 49v^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where equations (1) and (2) are, of course, both equations involving only numbers.

Dividing (2) by (1) we get

$$E = 10 \times 49;$$

and the unit of kin. en. is 490 faspen.

Note.—Not $\frac{1}{2} \times 10 \times 7^2$; but simply 10×7^2.

To return to example **D.**

Let T'', L cms, M grammes, K casgram, E casgrammen, be the units.

Then for the moving mass [in order to find the two forms of each of the equations $k=mv$, $\eta=\frac{1}{2}mv^2$;] we have as its

mass	velocity	momentum	kin. en.	
$10^5\div M$	$36\div\frac{L}{T}$	20	2	of these units;
10^5	36	$20K$	$2E$	of *C.G.S.* units;

whence the equations are

$$\begin{cases} 20 = \dfrac{10^5}{M}\times\dfrac{36T}{L}; & (3) \\ 20K = 10^5\times 36; & (4) \end{cases}$$

$$\begin{cases} 2 = \dfrac{1}{2}\cdot\dfrac{10^5}{M}\times\left(\dfrac{36T}{L}\right)^2; & (5) \\ 2E = \dfrac{1}{2}.10^5\times(36)^2; & (6) \end{cases}$$

From (4) and (6)

the unit of momentum $=180000$ casgram;
the unit of kin. ener. $=32400000$ casgrammen.

From (3) and (5)

the unit of mass $=1$ kilogramme;

$$\text{and } \frac{L}{T}=180 \qquad (7)$$

To determine the units of length and time we require the other datum.

In the case of the acceleration [to find the two forms of the equation $v=at$;] take t_1 seconds as a time during which the

acceleration of gravity is acting; the acquired velocity will be $981t_1$ cas; then we have as the

velocity	acceleration	time	
$981t_1 \div \frac{L}{T}$	10·9	$t_1 \div T$	of these units;
$981t_1$	981	t_1	of *C.G.S.* units;

whence the equations are

$$\begin{cases} 981t_1 \times \frac{T}{L} = 10{\cdot}9 \times \frac{t_1}{T}; & \quad (8) \\ 981t_1 = 981 \times t_1; & \quad (9) \end{cases}$$

whence $\frac{L}{T^2} = 90$ . . . . (10)

From (7) and (10)

the unit of length = 360 centimetres;
the unit of time = 2 seconds.

E. Under a mass-acceleration of 100 sfasp, a mass of $3\frac{3}{4}$ lbs. moves through 30 feet along a smooth horizontal plane. Find the velocity acquired; and the time taken.

We have
$$\frac{1}{2} mv^2 = ma \times x;$$
$$\therefore \frac{1}{2} \cdot \frac{15}{4} \cdot v^2 = 100 \times 30;$$
whence the velocity acquired = 40 fas.

Again,
$$mv = ma \times t;$$
$$\therefore \frac{15}{4} \cdot 40 = 100 \times t;$$

whence the time taken = $1\frac{1}{2}$ second.

F. Find the number of centimetres through which a mass of

2 kilogrammes will move in 2 seconds under a mass-acceleration of 25000 scasgram.

We have $$mx = \frac{1}{2} ma \times t^2;$$

$$\therefore\ 2000 \times x = \frac{1}{2} 25000 \times 4;$$

whence number of centimetres is 25.

EXAMPLES.—IV.

1. If 1 foot=30·4797 centimetres, and 1 pound=453·5926 grammes; how many casgram are there in a fasp?

2. Of two masses of 1 lb. and 8 lbs. moving in the same direction, the total momentum and the total kinetic energy are respectively 12 fasp and 12 faspen: find the velocities in fas.

3. The masses of a balloon and of the air displaced being respectively 2180 and 2360 kilogrammes; find the initial acceleration at a place where the acceleration of gravity is 981 scas.

4. Find the unit of time, when the units of length, mass, and mass-acceleration are 80 ft., 10 lbs., and 32 sfasp.

5. Under a mass-acceleration of a million scasgram, a mass of 3 kilogrammes moves through 15 metres. Find the velocity acquired, and the time taken.

6. What, in sfasp, must be the mass-acceleration when a mass of 12 lbs. moves through 2 inches in $\frac{1}{2}$ a second?

7. Two inelastic balls A and B, of masses respectively 3 and 2 kilogrammes, are moving in the same direction, the foremost, B, with a velocity of 300 cas. A impinges on B and doubles its [v. B?]

velocity. Find the velocity of A before the impact. [The momentum of the system remains the same. After the impact the velocities are equal.]

8. Three bodies are moving with velocities of 21, 14, and 7 fas; and the sum of their masses, the sum of their momenta, and the sum of their kinetic energies, are respectively 1 lb., 11 fasp, and $73\frac{1}{2}$ faspen; find the number of pounds in the mass of each body.

9. Taking 1 centimetre $=\frac{2}{5}$ inch, and 1 kilogramme $=2\frac{1}{5}$ lbs.; what decimal of a sfasp is a scasgram?

10. When the units of time, length, and mass-acceleration are 1 minute, 1 mile, and 22 sfasp; find the unit of mass.

11. Two masses move from rest under the same mass-acceleration, and describe the same length, one taking three times as long as the other. Compare their kinetic energies and their momenta at the end of their motion.

12. Under a mass-acceleration of a million scasgram, what mass will be moved through 9 metres in 3 seconds?

13. Two perfectly elastic balls, of masses 1 lb. and 12 oz., are moving in the same direction with velocities respectively of 35 and 28 fas; find the velocities after impact. [The momentum of the system remains the same; and so does the difference of the velocities; but the smaller ball is now the faster.]

14. When two perfectly elastic balls, moving in the same direction, collide, show that the sum of the kinetic energies remains constant. [The total momentum of the system and the difference of the velocities remain the same before and after the collision.]

15. If a ball of lead of specific density $11\frac{1}{2}$ be dropped in water at a place where the acceleration of gravity is $983\frac{1}{4}$ scas; find the initial acceleration.

16. Find in metres the unit of length when the units of time, mass, and mass-acceleration are 5″, $4\frac{1}{2}$ kilogrammes, and 450,000 scasgram.

17. A one-ounce mass, starting with a velocity of 47 yards per minute, has after $\frac{1}{20}$ second a velocity of 15 yards per minute; find the number of sfasp in the constant retardation.

18. In what time, under a mass-acceleration of 32·2 sfasp, will a mass of 1 lb. move through 1610 feet?

---

19. Two bodies, the sum of whose masses is 1 lb., are moving with velocities of 3 inches and 15 inches per second; and their total momentum is 1 fasp. Find the masses.

20. How many casgrammen, approximately, are there in a faspen; if 1 lb.=453·5926 grammes; and 1 foot=30·4797 centimetres?

21. A weightless and massless string is attached to a mass of 10 lbs. resting on a smooth horizontal table, and, passing over the edge, supports a mass of $\frac{1}{2}$ oz. Find the acceleration of the whole $160\frac{1}{2}$ oz. at a place where the mass-acceleration, due to gravity, of a falling mass of m lbs. is $32·1m$ sfasp.

22. Find in casgram, casgrammen, and scasgram respectively, the units of momentum, kinetic energy, and mass-acceleration; when the units of time, length, and mass are 1 minute, 1600 metres, and $6\frac{3}{4}$ kilogrammes.

23. If a particle, starting with a velocity of V cas, acquire an additional velocity of v cas after it has passed through l centi-

metres; and a further additional velocity of v cas when it has covered $3l$ centimetres from the starting; prove that $v=2V$.

24. A train of mass 105 tons, on a horizontal road, is subject to a *nett* mass-acceleration of 20,000 sfasp. How long will it take to go from rest ·539 of a mile; and how many miles per hour will it then be going?

---

25. Two perfectly elastic balls, of masses 600 and 400 grammes, are moving in the same direction with velocities respectively of 250 and 150 cas; find the velocities after impact. [The momentum of the system and the difference of the velocities remain the same after impact as they were before.]

26. If two masses of m and m' lbs. are moving at one time with velocities of u and u' fas; and at another time with velocities of v and v' fas; show that if the total momentum of the system is the same at each time, the total kinetic energy must, in general, be different.

27. The two ends of a weightless and massless string, passing over a pulley, hang vertically, and support masses of 550 and 540 grammes. Find the acceleration of the whole 1090 grammes, at a place where the mass-acceleration due to gravity of a falling mass of m grammes is $981m$ scasgram.

28. At a place where the acceleration of gravity is 32·12 sfas; find the units of time, length, mass, momentum, and kinetic energy; when a mass of 242 lbs. moving with a velocity of 5 fas, has 11 units of momentum and one-half the unit of kinetic energy; and when the acceleration of gravity is 2·92 units of acceleration.

29. Find the mass-acceleration of a mass of 20 kilogrammes which, in passing through $2\frac{1}{2}$ metres from rest, has gained a velocity of 50,000 cas.

30. A train of mass 10^5 kilogrammes on a horizontal road is subject to a nett mass-acceleration of 3×10^8 scasgram. How long will it take to go from rest $2\frac{2}{5}$ kilometres; and how many kilometres per hour will it then be going?

SECTION V.

THE FIRST LAW OF MOTION.

78. LAW 1.—**The Momentum in a Mass (or system of masses) cannot be increased or diminished except by the action of External Force.**

79. Hitherto we have dealt only with the motion in a mass (its momentum, its kinetic energy, and its mass-acceleration), not with the *cause* of the motion.

Change of *momentum* (the 1st Law tells us) must be caused by an agent external to the mass.

80. A "definition of force" often given is, "That which produces change of momentum is called force"; but, as a fact, this is but the 1st Law in another form.

It is not a definition of force. We are merely describing it in a periphrasis by its sensible effects.

it is a definitio nominalis not realis

81. Just as Euclid's Geometry is founded upon certain definitions and axioms, so is Dynamics.

Some of these definitions have already been given, and more will have to be given; and in this and the two suc-

ceeding Sections we have to learn the three axioms which connect force and motion.

82. It must be remembered that the properties of mass and force might have been such as to require a totally different set of axioms; so that these axioms must be considered as resting on convictions drawn from observation and experiment, *not* on intuitive perception.

this is not true for the first law

They are therefore usually called *Laws.*

83. It is important to notice that we are concerned in these three Sections with (in addition to velocity, time, length, force and work-done) *mass* only.

We have nothing to do as yet with *weight*; though, as weight is a simple kind of force, we may usefully employ it in illustrations.

Mass was considered in Section I.; momentum and kinetic energy in Section IV.; and now we have to determine the connection between these and force and labour-expended.

84. It will be readily allowed that, if a single heavy particle, or a mass like a cannon-ball, has no momentum, *external* force will be required to give it momentum.

85. And again that, if a complicated system of masses, such as a shell and its contents, has no momentum, force *of some kind* will be required to give it momentum.

The first law of motion goes further than this, and declares that *external* force only can give momentum. The explana-

tion of this in the case of the bursting of the shell (where momentum, at first sight, certainly seems to be produced by an *internal* force), will be given presently; and must be given before we understand the full meaning of the law.

86. Again, if a mass (as a shot) has some momentum already, it will be allowed to be a reasonable supposition that only external force can *increase* the momentum.

I should at once ask for the cause of this [illegible]

But it would be a natural supposition that, in course of time, momentum might be *lost*, waste away. But the law tells us that there cannot be any loss of momentum except by the action of external force.

87. To see that momentum in a moving rigid mass is not diminished except by the action of external force, we may notice that a cricket-ball comes to rest sooner on grass than on a road; and that a marble comes to rest sooner on a road than it does on the pavement; and comes to rest far sooner on a pavement than it does on ice.

These facts point to the truth that, were it not for the friction of the ground and air, a mass once set rolling or sliding would for ever retain its momentum. A curling-stone, for instance, will go a very long distance; being only gradually stopped by the friction of the ice and the resistance of the air.

88. Definition.—This purely negative but universal property of matter, viz., *that it cannot of itself modify its state of rest or motion*, is called *Inertia*.

Is this any more a defin. than that given of force in art. 80?

Thus it is advantageous to have a fly-wheel to a pump; to equalize the motion and so make the work easier. In the upstroke harder work is required than in the downstroke; in fact, in the downstroke the force of gravity will help to turn the wheel; then the inertia of the wheel will keep it rotating, and so help the upstroke.

It is to inertia again, for instance, that are chiefly due the terrible results of railway collisions.

89. It must not be supposed that the 1st law is entirely obvious.

Two main facts which impress themselves upon us are:

i. All momentum that we produce, as of a body thrown along the ground or of a rotating wheel, lasts for a certain time and then vanishes.

ii. On the other hand, the momentum of a falling body becomes greater and greater.

The true explanation of these facts is, of course, involved in the law of gravitation, and in the 1st and 2nd laws of motion. But the explanation formerly attempted was that a motion such as (i.) is a forced motion; and a motion such as (ii.) a natural motion; and that forced motions decay and tend to cease, while natural motions always increase.

These ideas were found untenable. "Forced motions," *e.g.*, are found to decay less and less as obstacles are diminished; so that there is evidently nothing in the "forced motion" itself to account for the gradual cessation.

90. The best proofs of the 1st law are necessarily indirect. We take the law and from it make some deduction.

If that deduction is true, we infer that there is truth in the law out of which it has been deduced.

Thus by a mathematical investigation it is established that if (as in a top) each part of a body is constrained to move in a circle, then the body will rotate with uniform velocity if the first law be true; but will not so rotate if the law be not true.

Our experience teaches us that such bodies do rotate uniformly.

91. We have now to return to the difficulty which was mentioned above, viz., as to the meaning we are to assign to such a statement as the following:

A shell is fired vertically upwards, and at the moment when it comes to rest it bursts; and, by the 1st law, the bursting force (not being an external force) does not increase the momentum of the shell.

92. It is obvious that individual pieces of the shell have very considerable increase of momentum, some in one direction and some in another.

What the law evidently teaches us is that, if the momentum in one direction is put against that in the opposite, the resulting total will be zero.

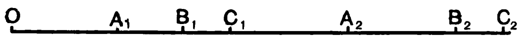
93. Now a convenient way of expressing this is to say that if the mass supposed collected at its "centre" has no momentum; *i.e.* if the "centre of mass" has no velocity; external force will be required to give it velocity.

94. It is to be noted that hitherto we have supposed the moving body to be a point.

No mass is really a point; and it would be extremely complicated to have to consider the motion of each separate particle of mass.

This we may avoid by means of a principle which enables us to fix upon a centre to the mass, and to consider only its motion.

95. Definition.—If a number of particles of equal mass lie along a straight line,



m_1 of them at A_1, B_1, C_1 . . . . ., and m_2 at A_2, B_2, C_2 . . . . ., then the *Centre of Mean Distances* is at a distance from O

$$\frac{OA_1+OB_1+\ldots\ldots+OA_2+OB_2+\ldots\ldots}{m_1+m_2}.$$

If now all the m_1 particles be at one point A_1, and all the m_2 particles be at A_2, the centre of mean distances is at a distance from O

$$\frac{m_1 OA_1+m_2 OA_2}{m_1+m_2}.$$

96. Let now $OA_1=x_1$ centimetres; $OA_2=x_2$ centimetres; and let the distance of the centre from O be $\bar{x}$ centimetres; then

$$\bar{x}=\frac{m_1x_1+m_2x_2}{m_1+m_2}.$$

If each particle have a mass of 1 gramme, then at A_1 there are collected m_1 grammes, and at A_2 there are col-

lected m_2 grammes; and the point given by $\bar{x}$ is now called the *centre of mass* of the $\overline{m_1+m_2}$ grammes.

97. Now suppose two particles of masses m_1 and m_2 grammes to be on a straight line, and to be moving uniformly in that straight line, away from O.

Let the particles and their centre be at one time at distances from O of x_1, x_2, and $\bar{x}$ cms.; and, after a time t'', at distances respectively of X_1, X_2, and $\bar{X}$ cms.;

Then X_1-x_1, X_2-x_2, $\bar{X}-\bar{x}$, are the cms. travelled in the time t'';

Let v_1, v_2, and $\bar{v}$ cas be the corresponding velocities;

$$\text{then } \bar{x}=\frac{m_1x_1+m_2x_2}{m_1+m_2};$$

$$\text{and } \bar{X}=\frac{m_1X_1+m_2X_2}{m_1+m_2};$$

$$\therefore \bar{X}-\bar{x}=\frac{m_1(X_1-x_1)+m_2(X_2-x_2)}{m_1+m_2};$$

$$\therefore \bar{v}t=\frac{m_1v_1t+m_2v_2t}{m_1+m_2};$$

$$\therefore (m_1+m_2)\bar{v}=m_1v_1+m_2v_2;$$

or the momentum of the whole mass supposed collected at its centre is equal to the sum of the momenta of the two parts.

98. As the sum of the momenta of the two parts is the total momentum of the system; and there are no external forces, and therefore the sum of the momenta remains constant; therefore so also does the momentum of the whole mass supposed collected at its centre.

99. A similar result maintains for any number of particles in a straight line; and can be easily proved.

100. Also by the aid of co-ordinate geometry we may obtain similar expressions for the centre of mass of any number of particles lying in a plane; or in space; and therefore of any mass whatever.

101. DEFINITION.—The *Centre of Mass* of any system of particles is a point whose distance *from any plane* is thus determined:

If m_1, m_2 . . . grammes be the masses of the particles;
x_1, x_2 . . . . cms. their distances from the plane;
and $\bar{x}$ cms. the distance from the plane of the C. of M.;

$$\text{then } \bar{x} = \frac{m_1x_1 + m_2x_2 + \dots\dots}{m_1 + m_2 + \dots\dots}.$$

102. From this we get, as before,

$$\bar{v}(m_1 + m_2 + \dots\dots) = m_1v_1 + m_2v_2 + \dots\dots$$

103. The result is that we may now thus read the first law:

The centre of mass of a moving system, free from external force, is either at rest or moving uniformly in a straight line.

104. There is a class of cases which at first sight seems to refute the statement that the centre of a mass cannot move except by the application of external force.

But in all cases of change of motion it will be found that there is an external force. It may not be the actual motive power, but it is necessary for the motion.

An engine (in which, of course, the force is wholly internal) could not move itself or pull a train, but for the friction between the wheels and line.

So, generally, whenever a mass is in contact with another, *i.e.* when there is a *point d'appui*, an internal force may cause motion.

What is internal & what external force?

105. In the equation

$$(m_1+m_2)\bar{v}=m_1v_1+m_2v_2,$$

for two masses moving in a straight line, we supposed the masses to be moving in the same direction.

The same equation will serve us if the masses are moving in opposite directions; provided that we give a negative sign to v_2.

106. Thus, if two inelastic balls, of m_1 and m_2 lbs., moving towards one another with velocities of v_1 and v_2 fas, come into collision; it is clear that, in many cases, both lose velocity; and Newton's experiments teach us that they will move together, with equal velocity, in the direction in which there was, beforehand, the greater momentum.

But, as there is no external force, the total momentum must be the same after as before the collision; so that, if V fas be the common velocity after the collision, we must have

$$m_1V+m_2V=m_1v_1+m_2v_2.$$

This is impossible as it stands, as V is less than either v_1 or v_2; but if we give to v_2 the negative sign arising from its direction, we get the perfectly intelligible equation

$$(m_1+m_2)V=m_1v_1-m_2v_2.$$

107. So, generally, if there be any number of moving heavy particles, the sum of their momenta in any direction (when proper signs are given to the velocities) cannot be increased or diminished except by the action of external force.

If a shell, *e.g.*, explode when at rest in mid-air, the sum of the momenta of the pieces in any and every direction is zero.

108. As a particular case we get the very important theorem, that:

If two masses moving in a straight line come into collision (there being no external force), *the total momentum remains constant.*

109. This fact in the impact of masses is over and over again shown in Newton's experiments; thus verifying the basis (*i.e.* the 1st law) on which the theorem has been built up.

110. An extension of the 1st law of motion, consequent upon it, is this:

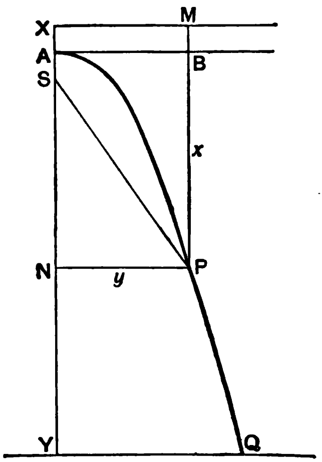
The momentum which a mass has in a straight line will not be altered by a force which is entirely at right angles to the straight line.

111. If a barge is to be towed along a canal the necessary forces are the pull of a rope from the bank, and the strain of

the water against the barge and rudder. If a second horse were on the bank over-against the barge, so that his pull were at right angles to the barge's way, he would simply pull the barge towards the bank, and would not affect the velocity in the direction of the canal.

So if a bullet be fired horizontally, it will, neglecting the resistance of the air, reach the ground in the same time as a bullet dropped at the moment of firing.

112. Suppose a mass of m grammes to be projected from A, the top of a tower AY (X cms. high), with a velocity of V cas in a horizontal direction.



Let APQ be the curve described; the resistance of the air being neglected.

If P be the position of the mass at the end of t seconds; then BP ($=x$ cms.) will be the vertical distance through which it has fallen owing to the acceleration of gravity;

PN ($=y$ cms.) will be the horizontal distance through which it has moved owing to its initial velocity.

By the extension of the 1st law (given above), the horizontal velocity will remain intact;

$$\therefore\ y=Vt; \quad \ldots\ldots\ldots \quad \text{(A)}$$

and by equation (3) in Section III.,

$$x=\frac{1}{2}gt^2; \quad \ldots\ldots\ldots \quad \text{(B)}$$

which two equations give us the position of the mass at any time.

113. If we eliminate t we get

$$\begin{aligned} y^2 &= V^2t^2; \\ &= V^2\frac{2x}{g}; \\ &= 4\frac{V^2}{2g}x; \quad \ldots\ldots\ldots \quad \text{(C)} \end{aligned}$$

which equation students of analytical conics will recognise as that of a parabola whose vertex is at A, axis along AN, and latus rectum $\frac{2V^2}{g}$.

114. If we take $AS=\frac{V^2}{2g}$; then the equation becomes

$$PN^2=4\times AS\times AN; \quad \ldots\ldots \quad \text{(D)}$$

the well-known property of the parabola whose focus is at S.

115. So that the path of a particle projected horizontally is a parabola.

116. Further examples of projectiles will be given in a subsequent Section.

WORKED EXAMPLES.

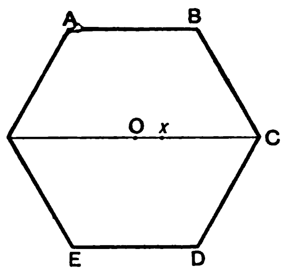
A. If a road make a sudden turn, a carriage passing too rapidly will be overturned. To which side, and why?

Towards the outer side, because of the want of force to change the direction of motion.

B. Why does a rider fall over the head of his horse when it suddenly stops; and fall backward when it suddenly starts?

In the former case his *inertia* keeps him moving though the horse stops; and in the latter case his *inertia*, again, keeps him at rest though the horse moves.

C. Find the distance from the centre of a regular hexagon of the centre of mean distances of equal particles at five of the angular points.



If the five points taken be A, B, C, D, E, it is easily seen that the centre of mean distances of A, B, D, E is O, the centre of the hexagon;

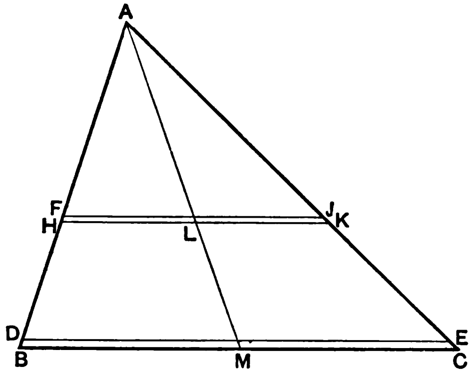
We may, therefore, consider 4 particles collected at O, and 1 at C;

$$\therefore\ Ox = \frac{4 \times 0 + 1 \times OC}{4+1};$$

$$= \frac{OC}{5};$$

$$= \frac{1}{5}\ \text{(side of hexagon)}.$$

D. Find the centre of mass of an uniform triangular lamina *ABC*.



A lamina is a very thin slice.

Let the density, with British units, be d.

Divide up the lamina into a very large number, n, of parallelopipedons (of which *DBCE* is one, and *FHKJ* another) with their long edges parallel to *BC*, and their widths (perpendicular to *BC*) all equal to one-nth of the perpendicular from *A* to *BC*.

[A parallelopipedon is a rectilinear solid which has all its edges parallel to one or other of three given straight lines; and, of course, *DBCE* will not be a true parallelopipedon; but, by taking its thickness very small, we may in the end eliminate any error arising from this source.]

If, now, a sq. ft. be the area of a section of either of the parallelopipedons perpendicular to its length;

$$\text{the mass of } DBCE = ad \times BC;$$

$$\text{and the mass of } FHKJ = ad \times HK;$$

$$= ad \times \frac{AL}{AM} BC;$$

where *ALM* is the line bisecting the base.

ALM clearly bisects HK and all other lines parallel to BC; and the centre of mass must lie in AM.

If G be its position we evidently have

$$AG = \frac{\text{sum of all such expressions as } ad \times \frac{AL}{AM} BC \times AL}{\text{sum of all such expressions as } ad \times \frac{AL}{AM} BC};$$

$$= \frac{\text{sum of all such expr. as } AL \times AL}{\text{sum of all such expr. as } AL};$$

$$= \frac{\left(\frac{AM}{n}\right)^2 + \left(\frac{2AM}{n}\right)^2 + \left(\frac{3AM}{n}\right)^2 + \ . \ . \ . \ + \left(\frac{n.AM}{n}\right)^2}{\frac{AM}{n} + \frac{2AM}{n} + \frac{3AM}{n} + \ . \ . \ . \ + \frac{n.AM}{n}};$$

$$= \frac{AM}{n} \cdot \frac{1^2 + 2^2 + 3^2 + \ . \ . \ . \ + n^2}{1 + 2 + 3 + \ . \ . \ . \ + n};$$

$$= \frac{AM}{n} \cdot \frac{n(n+1)(2n+1)}{6} \div \frac{n(n+1)}{2};$$

$$= \frac{AM}{n} \cdot \frac{2n+1}{3};$$

$$= \frac{2AM}{3} + \frac{AM}{3n};$$

When n is very large, the latter of these two expressions is to be neglected. It is very small, and its omission helps to balance the error which arises from taking the strips as exact parallelopipedons.

$$\therefore \ AG = \frac{2}{3} AM;$$

or the centre of mass of the triangular lamina is on the line from the vertex bisecting the base (called the *median*), at a distance from the vertex of two-thirds of the length of the median.

E. The head of a punt is close to shore, the punt being at right angles to the shore; and the occupant who is at the other end walks towards the shore. Why does the punt slip away from the bank? Will it continue to move?

The punt moves because the occupant in shifting his place tends to disturb the centre of mass of the system, which no *internal* force can move.

The motion being only motion of the parts of the system among themselves, there is no motion of the centre of mass; so the motion of the punt will not continue.

F. Why can a man rise from a chair, seeing that his centre of mass has to be moved forward?

He has a *point d'appui*; so he is able to move his centre of mass forward; and thus to rise. and. 104

G. Two inelastic balls, of mass 2 lbs. and $10\frac{2}{3}$ oz., move in opposite directions with velocities respectively of 200 and 300 fas; find the common velocity after impact.

If V fas be a common velocity

$$2V+\frac{2}{3}V=2\times 200-\frac{2}{3}\times 300$$

(the negative sign being taken because the $10\frac{2}{3}$ oz. is moving in opposite direction to the 2 lbs.);

whence the velocity is 75 fas.

H. It is said that "If a stone be dropped from the top of the mast of a ship in motion, the stone will fall at the foot of the mast notwithstanding the motion of the ship." Show that, if we neglect the resistance of the air and the rotation of the earth, this is a necessary consequence of the 1st law.

No force is acting on the stone which can interfere with the velocity it already has in the direction of the ship's motion.

+ I. From the top of a tower, of height 100 feet, a particle is projected horizontally with the velocity that would be acquired by falling 900 feet. Find the range on the horizontal plane through the foot of the tower.

In the figure in Paragraph **112**, YQ is the range $=R$ feet, $AY=100$ ft.

From equation (4) of Section III,

$$\text{horiz. vel.} = \sqrt{2g \times 900} \text{ fas};$$
$$=30\sqrt{2g} \text{ fas.}$$

Hence $$R=30\sqrt{2g}\times t;$$

and $100=\frac{1}{2}g\times t^2$ (from equation (3) Section III.);

whence the range $=600$ feet.

EXAMPLES.—V.

1. Why is it dangerous to alight from a train in motion?

2. Explain why, at a railway station, a horse has so much difficulty in starting the couple of carriages which it has to shunt; and so much ease in drawing them afterwards.

3. Show that we obtain a similar value for the mean distance of a number of particles which lie along a line, whatever the point from which the distances are measured.

4. Show that the centre of mass of any number of particles moving uniformly in a straight line also itself moves uniformly.

5. A fly is at one end of a match ($2\frac{1}{4}$ ins. long) floating on water. If the mass of the fly is one-fifth that of the match, what distance will the match move if the fly walk to the other end? [Neglect the resistance of the water.]

6. How does a boy, standing on a swing, swing himself higher and higher?

7. Two inelastic balls, one of 900 grammes, and the other of 300 grammes, move in opposite directions with velocities respectively of 6 and 9 cas. Find the common velocity after impact.

8. If a book be dropped or thrown in a railway carriage, explain why the velocity of the train has no apparent effect on the motion of the book. Has it any real effect?

9. At a place where the acceleration of gravity is 32·25 sfas, a particle is projected horizontally with a velocity of 129 fas from a point whose height above the horizontal plane is 258 feet. Show that the focus of the parabola is in the plane; and find the angle at which the particle strikes the plane.

---

10. Explain the action of throwing a quoit.

11. Explain why a bullet fired against a door will perforate the door without moving it.

12. Find the centre of mean distances of 60 points along a straight line Ox; the 1st being at 1 inch from O, the 2nd at 3 ins., the 3rd at 5 ins., and so on.

13. Show that the centre of mass of three equal particles, placed at the vertices of a plane triangle, is at the same point as the centre of mass of the triangle considered as an uniform triangular lamina.

14. When a punter has run from one end to the other of his punt with his pole in the ground; show that if he returns quickly to the former end he materially lessens the velocity.

15. When a man stumbles, how is he able to save himself from falling; seeing that his centre of mass is thrown so far forward? and why is it so much more difficult on ice?

16. Two bodies having equal momenta move in opposite directions and collide; show that the momenta will be equal after impact.

17. Explain why two books, one of which is thrown horizontally and the other dropped, reach the ground in the same time.

18. In the example in Paragraph **112**, show, without assuming the curve to be a parabola, that $SP=PM$; when BM a continuation of PB is equal to $\frac{V^2}{2g}$.

---

19. If a billiard-ball be gently pushed, why does it gradually come to rest?

20. Place a playing-card upon the back of the fist; put a shilling on it; and flick the card with the finger (taking care to flick horizontally). The card will be driven away, but the coin will remain on the fist. Explain this.

21. Find the distance from the vertex of a square whose diagonal is 2 yards, of the centre of mean distances of the three other vertices.

22. The intersection of the diagonals of a uniform quadrilateral lamina is one of the points of trisection of each diagonal. Show that the centre of mass of the quadrilateral is on the line joining the remaining point of trisection of one diagonal with the point of bisection of the other. [Use qu. 13.]

23. A frog is at one end of a board 1 foot long whose mass is twice that of the frog, and which is floating in water. Neglecting the resistance of the water, find how far the board will move if the frog jumps to the other end.

24. When a shell bursts on the ground how is the apparent increase of momentum accounted for?

25. Two inelastic masses of 200 and 100 kilogrammes collide, and both come to rest. If the second was moving at the rate of a mile a minute; with how many fas was the first moving?

26. When a particle slides on a smooth surface under the action of no external force, show that its velocity is constant.

27. Find the range on the horizontal plane of a mass projected horizontally, with the velocity of V fas, from the top of a tower h feet high, at a place where the acceleration of gravity is g sfas.

---

28. When a railroad lies on a curve, one rail is laid higher than the other. Which? and why?

29. If a basin of water be placed in a toy wagon, and the wagon be moved forward, the water will rise up. In which direction? and why?

30. Find the distance from the vertex of a cube, whose side is 1 foot, of the centre of mean distances of the other seven vertices. [Take $\sqrt{3}=1\frac{3}{4}$.]

31. Find the centre of mass of four uniform straight rods, one of which is the diameter of a circle, and the others chords parallel to and on the same side of the diameter and subtending at the centre angles of 120°, 90°, and 60°.

14. When a punte[r] ... his pole ... [co]unt with ... quickly to the former ...

15. When a man s... from falling; seeing t... forward? and why is it ...

16. Two bodies havin[g] ... directions and collide; sh[ow] ... after impact.

17. Explain why two ... zontally and the other dr[opped] ... time.

18. In the example in Par[t] ... the curve to be a parabola, t[he] ... tinuation of PB is equal to $\frac{V^2}{2g}$

19. If a billiard-ball be gentl[y] ... come to rest?

20. Place a playing-card upo[n] ... shilling on it; and flick the card ... flick horizontally). The card will ... will remain on the fist. Explain th[is].

21. Find the distance from the v[ertex] ... diagonal is 2 yards, of the centre of ... other vertices.

22. The intersection of the diagon[als] ... ateral lamina is one of the points of tris[ection] ... how that the centre of mass of the qua[drilateral] ... joining the remaining point of trisectio[n] ... the point of bisection of the other. [Use

... lengthways down a stream
... has at its upper end a dog
... bank. In what time must
... when he gets there, the

... a room sitting on a chair,

... masses 3 lbs. and 1 lb., are
... respectively of 10 fas
... impact. [The algebraical
... before and after.] cf. Ex. 13 p. 42

... walk along tossing a

... metres high, a mass is
... 1,000 cas, at a place
... find in metres the
... foot of the tower;
... with which the

... and Drakes."

... handle of a hammer
... more firmly?

... radius is 3 yards is
... from the centre
... of the points

... triangular laminæ

on the same base BC and on the same side of it. If the centre of mass of the four-sided figure $ABDC$ is at D; find the ratio of the areas of the two triangles.

41. Two children, of mass 3 and 4 stone, are one at each end of a plank of length 20 feet, which is lying on perfectly smooth ice. What is the ratio of the velocities with which they must walk towards the middle of the plank that the plank may not move? which of the two will get first to middle? and how far, then, will the other have gone?

42. Why is a railway engine able to move though no external force seems to act upon it?

Two inelastic balls are moving towards one another with velocities one 4 times the other; and, by collision, the swifter loses one-fourth of its momentum; find the relative masses of the balls. [After collision two inelastic balls move with equal velocities.]

Explain why a circus-rider alights on her horse after jumping and going through a hoop.

At a place where the acceleration of gravity is 32·25 sfas, a ball is projected horizontally from the top of a tower and strikes the horizontal plane through the foot of the tower. The range is 645 feet, and the latus-rectum of the parabola ... find the height of the tower and the velocity of

SECTION VI.

THE SECOND LAW OF MOTION.

117. LAW 2.—**When Momentum is produced, it is by the action of Force; and the amount of Momentum produced in a given Time is Proportional to and in the Direction of the Force.**

118. The difference between the 1st and 2nd laws is, that in the 1st law we only speak vaguely of force and momentum without finding any but a negative connection between them; while in the 2nd law we determine this connection.

119. Under the 1st law we have been enabled to obtain results in cases where no external force is acting; or where the external force is such as not to affect the motion.

By law 2 we are enabled to investigate the result in a mass, of the action of external force which interferes with the state of rest or motion.

We may say that the 1st and 2nd laws together declare that it is force alone which can produce a change of momentum; and tell us how the change of momentum depends on the magnitude and the direction of the force.

120. The British Unit of force is obviously that force which in 1″ produces 1 fasp.

The *C.G.S.* Unit of force is that which in 1″ produces 1 casgram.

121. It is necessary to introduce "time" into the law, for two reasons:

1. *Because the* AMOUNT *of time is a material element.*

It is clear that if, for instance, a casgram were produced in half a second, the force must have been double of the *C.G.S.* unit-force.

And again, if it takes a minute to produce a fasp, the force acting is but the sixtieth of the British unit.

2. *To show that it is the* TIME *and* NOT *the* LENGTH *during which the momentum is being produced, by which force is to be measured.*

If, during a certain time two forces, of p and p' British units of force, produce respectively k and k' fasp; then

$$p : p' :: k : k'.$$

But if, during a certain *length* two forces of p and p' British units produce k and k' fasp, this equation is no longer true.

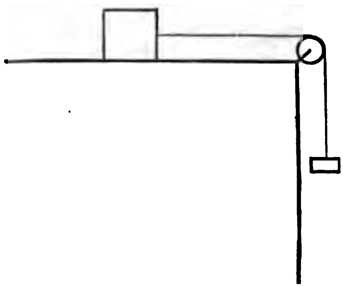
What is true is (as we shall see by the 3rd law) that if during the passage over a certain length two forces of p and p' British units produce η and η' faspen; then

$$p : p' :: \eta : \eta'.$$

122. The 2nd law, which at first sight seems not unreasonable, and which is really often assumed without

adequate explanation, is really not so obvious when we come to critically examine it.

123. Imagine a perfectly smooth table with a perfectly frictionless pulley at the edge; place a *mass* M on the table, and connect it, by a thin massless string which passes over the pulley, with a smaller *mass* m hanging vertically.



Now it is clear that the attraction of the earth on m is the moving force of the system; but that the attraction of the earth on M cannot affect the motion.

Yet the *mass* of M does affect the motion.

124. At first sight we should be inclined to say that, with a perfectly smooth table and pulley, the mass of M can make no difference to the motion; as its weight cannot influence the movement; and there is no friction.

But the 2nd law assures us that this idea is wrong.

For $M+m$ being the number of lbs. of mass in the system;
and v the number of fas at any time;
then $(M+m)v$ is the number of fasp at the same time;

And the law tells us that the number of units of force required to produce this momentum is proportional to $(M+m)v$;

And so if the hanging mass remains constant, *i.e.* if the

force acting on the system remains constant, then $(M+m)v$ remains constant; and so if M be increased the velocity will be diminished.

125. We at once acknowledge the truth of the converse of this proposition, viz., that the effort required to stop, in a given time, a mass moving with a certain velocity depends on the mass. A man who could easily, and in a short time, stop a cricket ball moving at the rate of 30 miles an hour, would avail nothing against a train moving at that rate; although in the case of the train he would have friction and considerable air-resistance helping him.

126. It is as impossible to *prove* the laws of motion by any simple direct means, as it is to prove the law of gravitation.

The law of gravitation declares that every particle in the universe attracts every other particle with a force varying inversely as the square of the distance between them. It is only when we come to examine the motion of the planets, and to see how this simple law explains their every movement, that we perceive the complete truth of the gravitation law.

So, until we have built up a Dynamical Science, and have found that the whole theory of motion hangs together in a way in which no theory could hang together which was founded on false premises, we cannot feel that we have a complete proof of the truth of the elementary "Laws of Motion."

127. We must not be discouraged because of this initial difficulty.

It is necessary to insist upon it, and to point out the difficulty; because, too often, the laws of motion are set down as being easy of comprehension, the learner takes them as being easy to grasp, fails to examine into them properly, never tries any illustrations for himself; and then, afterwards, is really discouraged because he fails in problems which a true understanding of the laws would have enabled him to solve.

The illustration given above (of the mass on the smooth table) is one in point. There are many who would have to confess that, on first trying the problem, they omitted all reference to the mass of *M*.

128. The laws cannot be entirely obvious, or it would not have been left to modern philosophers to formulate them. Some of the great minds of antiquity would certainly have seen the truths involved.

Many important truths were discovered of old time; the effect of atmospheric refraction, *e.g.*, was known to Ptolemy; the geometrical treatise of Euclid was written B.C. 300; but the idea of inertia was first recognised by Galileo (born A.D. 1564); and before his time the principles of mechanics were altogether faulty. That "circular motion is perfect" was a ruling idea. Archimedes even completely failed in his attempt at Dynamics.

129. In order to logically establish the truth of the 2nd law it would be necessary to prove that, in every instance,

1*stly*, with a given force, acting for a given time, the greater the mass the less the velocity produced;

2*ndly*, with different forces, acting for the same time, the greater the force the greater the momentum.

Then by Euclid, Book v., Definition 5, we should deduce that,

1*stly*, with a given force, acting for a given time, the velocity is *inversely* proportional to the mass;

2*ndly*, with different forces, acting for the same time, the change of momentum is directly proportional to the force.

The 1st of these would show that it is momentum which a force produces; the 2nd that the momentum produced in a given time is proportional to the force.

130. It is obvious that all that we can here expect in the way of proof is a few illustrations to show the truth of the law in some simple cases.

131. Thus, to show that, with a given force, the greater the mass the less the velocity; we may by means of a string hang a kitchen "weight" over the edge of a smooth table; this will serve as the given constant force. At the other end of the string, and resting on the table, will be the mass on which we are experimenting. It will be easily seen that the greater the mass the less the velocity.

132. And to show that the greater the force the greater the momentum, we may take such simple instances as the blow of a cricket bat; or the throwing of a stone.

133. Further instances will be found in the examples at the end of this section.

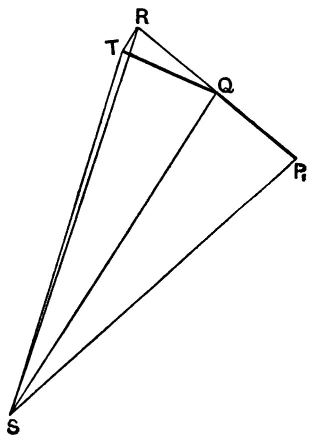
If any one could bring forward an instance in which the law does not hold, that would be fatal to the truth of the law. This, of course, is impossible.

134. But, after all, as before remarked, the real way in which the truth of these laws has been established is an indirect one.

Results derived from the laws, when tested, are always found to be true.

The following is an illustration of this.

135. PROBLEM.—*Suppose two planets to revolve round the Sun in circles of radii* a_1 *and* a_2 *feet, in respectively* T_1 *and* T_2 *moments;*[1] *then the attractive force of the Sun being inversely as the square of the distance, find the relation between the radii and the times.*



Let S be the Sun, and P_1 be one of the planets.

[To avoid the difficulty connected with the curve, we shall assume the path of the planet during each moment to be a straight line.]

Suppose it in one moment to pass from P_1 to Q.*

[We also suppose the action

[1] A moment is a very small but definite interval of time.

* Of course, in the figure the scale of P_1Q is enormous compared with that of SP_1; and the angles RQT, QSP are, in consequence, enormously exaggerated.

of the Sun to take place in jerks at the end of each moment.]

Now, by the 1st law, if there were no jerk at Q the planet would retain its motion, and would simply move in the same straight line with the same velocity;
and so at the end of another moment would arrive at R, off the circle,

$$\text{where } QR = P_1Q.$$

But by reason of the jerk it, as a fact, arrives at the end of this second moment at T on the circle.

Now, by the 2nd law, RT must be parallel to SQ; because the whole of the new motion given to the planet must be in the direction of the force.

Therefore, the displacement arising from the jerk

$$\begin{aligned} &= RT; \\ &= RQ \times \frac{QP_1}{P_1S};* \\ &= \frac{QP_1^2}{P_1S}. \end{aligned}$$

Now as QP_1 is described in a moment, and there are T_1 moments in a complete round;

$$\begin{aligned} \therefore\ QP_1 \times T_1 &= \text{whole circumference}; \\ &= 2\pi \times P_1S; \end{aligned}$$

$$\therefore \text{ displacement due to jerk} = \frac{4\pi^2 a_1}{T_1^2} \text{ feet.}$$

But this displacement is, of course, proportional to the velocity impressed by the Sun; *i.e.* proportional to the

* The triangles TQR, P_1SQ are similar; for they are isosceles, and $\angle\ TQR = \angle\ P_1SQ$.

momentum impressed; i.e. *by the 2nd law*, proportional to the impressing force; i.e. *by the data*, inversely proportional to the square of the distance;

$$i.e.\ \frac{\text{displacement in the case of planet } P_1}{\text{displacement in the case of planet } P_2} = \frac{\frac{1}{a_1^2}}{\frac{1}{a_2^2}};$$

$$\therefore \frac{4\pi^2 a_1}{T_1^2} : \frac{4\pi^2 a_2}{T_2^2} :: \frac{1}{a_1^2} : \frac{1}{a_2^2};$$

$$\therefore T_1^2 : T_2^2 :: a_1^3 : a_2^3.$$

136. Now in testing this, we have to remember that the planets do not revolve exactly in circles round the Sun; the distance of the Earth from the Sun, for instance, varying, according to the time of year, from 91 to 94 million of miles.

But if we take the mean distances, Kepler found by observation that *the squares of the periodic times are proportional to the cubes of the mean distances.*

Therefore so far we have a corroboration of the 2nd law.

137. The 2nd law comes into this problem mainly in finding the relative value of RT.

By the law, RT is proportional to the force; and, therefore, assuming the law of gravitation, to $\frac{1}{a^2}$.

On any other hypothesis we should have obtained an incorrect result.

Supposing for instance that the 2nd law had been thus

put: The kinetic energy produced in a given time is proportional to the force;
then we should have

$$\frac{4\pi^2 a_1}{T_1^2} : \frac{4\pi^2 a_2}{T_2^2} :: \frac{1}{2}\left(\frac{1}{a_1^2}\right)^2 : \frac{1}{2}\left(\frac{1}{a_2^2}\right)^2;$$

which would lead to the result

$$T_1^2 : T_2^2 :: a_1^5 : a_2^5;$$

which is not true.

WORKED EXAMPLES.

A. If the units of velocity, acceleration, and force, be respectively a furlong a minute, $\frac{50}{73}$ of the acceleration of gravity (which is 32·12 sfas), and 32·12 poundals;* find the units of time and mass.

Let T'' and M lbs. be the units;

then Unit Velocity = 1 furlong per 1′;
= 11 feet per 1″;

and Unit Force $\left\{\begin{matrix}\text{is that which}\\ \text{produces in}\end{matrix}\right\}$ M lbs. in T'' a vel. of 11 ft. per 1″;

i.e. „ M „ 1″ „ „ $\frac{11}{T}$ „ 1″;

i.e. „ M lbs. an acceleration of $\frac{11}{T}$ sfas;

but Unit Force „ M lbs. the unit acceleration;

$$\therefore \frac{11}{T} = \frac{50}{73}(32{\cdot}12);$$

∴ Unit Time is ½ a second.

---

* The British unit of force is called a Poundal.
The *C.G.S.* „ „ Dyne.

Again,

unit force	is that	which pro. in	M	lbs. in	1″	a vel. of	$\frac{11}{T}$	ft.	per	1″;
i.e.	,,	,,	M	,,	1″	,,	22	,,		1″;
i.e.	,,	,,	1	,,	1″	,,	$22M$	,,		1″;
but $\left\{\begin{matrix}\text{the British}\\ \text{unit force}\end{matrix}\right\}$	,,	,,	1	,,	1″	,,	1	,,		1″;

$$\therefore\ 22\,M = 32{\cdot}12\,;$$

$\therefore$ Unit Mass is 1·46 lbs.

---

i.e. in area **B.** Show that of all equal triangles the altitudes are inversely proportional to the bases.

It is easily seen from Euclid I. 40 that in every case, when two triangles are equal, that which has the greater altitude has the less base; and *vice versâ.*

Therefore the conditions in Euclid V. Def. 5 are fully satisfied; but inversely.

C. If a large anvil, resting on the ground, be struck with a sledge-hammer, little or no impression will be made in the earth; but if the anvil be only a mass of a pound or two, it will go a good way into the ground. Explain this.

The blow being in each case the same, the momentum imparted is, by the 2nd law, the same; therefore when the mass is small the velocity is large; and *vice versâ.*

[It is clear that here the attraction of the earth is (if *weight* had anything to do with the result) all in favour of a result opposite to that which is, in fact, the true one.]

D. Take two pulleys (*e.g.* those at the end of brass curtain rods), and fix them so that the axis of each pulley is quite

horizontal, and well oil the axis. Over pulley A pass a thin strong string, and suspend at either end of it masses of $\frac{1}{2}$ lb. and $6\frac{1}{2}$ oz.; and over pulley B pass a string with masses 2 lb. and 26 oz. When the larger mass is in each case brought near the top and let go, the two systems will be found to have the same motion. Show that this illustrates the 2nd law.

For, in A, the whole mass moved is

$$\tfrac{1}{2}\text{ lb.} + 6\tfrac{1}{2}\text{ oz.} = 14\tfrac{1}{2}\text{ oz.};$$

while the available force is the weight of

$$\tfrac{1}{2}\text{ lb.} - 6\tfrac{1}{2}\text{ oz.} = 1\tfrac{1}{2}\text{ oz.}$$

In B, the whole mass moved is

$$2\text{ lb.} + 26\text{ oz.} = 58\text{ oz.};$$

while the available force is the weight of

$$2\text{ lb.} - 26\text{ oz.} = 6\text{ oz.}$$

Then, since $\quad 14\tfrac{1}{2} : 58 :: 1\tfrac{1}{2} : 6,$

the numbers of units of mass are proportional to the numbers of units of force;

$\therefore$ by the 2nd law, the numbers of units of velocity (acquired in the same time) must be the same.

[It is interesting to note the much greater velocity with which A descends, when even only $\frac{1}{2}$ oz. is taken from the $6\frac{1}{2}$ oz.]

E. To what mass can a pressure producing 112 g sfasp acting for one minute, give a velocity of 65·7 miles per hour; at a place where g is 32·12?

The no. of fasp prod. in $1'' = 112 \times 32{\cdot}12$;

$\therefore$ no. of fasp prod. in $1' = 112 \times 32{\cdot}12 \times 60$;

Now 65·7 miles per hour$=\dfrac{219\times 11}{25}$ fas;

$\therefore$ no. of lbs. of mass$=\dfrac{112\times 32{\cdot}12\times 60\times 25}{219\times 11}$;

$\therefore$ required mass is 1 ton.

F. Examine by the 2nd law the statement that (apart from the resistance of the air) all bodies falling freely have the same motion.

The acceleration of gravity being g sfas, a mass of m lbs. will move with a mass-acceleration of mg sfasp [76]; *i.e.* will be acted on by a force of mg poundals [120];

$\therefore$ its momentum (mv fasp) at the end of t'' will be given by

$$mv = mg\times t \text{ [117]};$$

$$\therefore\ v = gt;$$

or the motion is independent of the mass.

G. Saturn being, approximately, 10 times as far off from the Sun as is the Earth; find the length of Saturn's year.

$$T^2 : (365\tfrac{1}{4})^2 :: 10^3 : 1^3;$$

whence Saturn's year$=$(approximately) 11,550 days. *ans. 135*

Examples.—VI.

1\. Compare the British and *C.G.S.* units of force. [Take 1 cm.$=\frac{2}{5}$ inch; 1 kilogramme$=2\frac{1}{5}$ lbs.]

2\. Prove that the areas of triangles of equal altitudes are to one another as the bases of the triangles.

3\. Explain this statement: "If a stone be large enough, I might lay it on my breast and suffer you to strike it with a

sledge-hammer with all your strength, without pain or risk. But if the stone were but a mass of a pound or two, your blow would probably kill me."

4. When in pulley A, in worked question **D**, the masses were 1 lb. and 12 oz.; and in pulley B 1 lb. and 13 oz.; it was found that when the 1 lb. in pulley A was stopped after going a certain distance, the 1 lb. in pulley B had travelled about $\frac{3}{4}$ of the distance. Show that this is in accordance with the 2nd law; and find the exact proportion of distance passed over.

5. If 352 poundals act for 28 seconds on a mass of a ton; at how many miles per hour will it then be moving?

6. Cut out a round piece of paper just a little smaller than a florin; place it on the florin held in a horizontal position; and let the two drop; they will come to the ground together. Explain this.

7. The mean distances from the Sun of Mercury and Uranus being respectively 35 and $1753\frac{1}{2}$ million miles; and the time of revolution of the former round the Sun being 88 days; what, approximately, is the time of the latter's revolution?

---

8. If the units of length, time and force be respectively a metre, a minute, and that which in 1 second produces 1000 casgram; find the unit of mass.

9. Prove that the volumes of cones of equal altitude are to one another as the bases of the cones.

10. Take a very light cardboard box (a tea-sample box, *e.g.*), hold it lightly between two fingers of the left hand, and drop into it an ounce "weight." There will be a pull on the fingers. Now fill the box (with coins, *e.g.*); then drop the ounce from a

corresponding height. Why is there, now, less pull on the fingers?

11. Show how the "Try your Strength" seen at a fair illustrates the 2nd law. A blow is given, and the strength is estimated by the height reached by the pointer.

12. Find the number of dynes of resistance (supposed constant) offered by a medium, when a mass of 1 kilogramme projected in it with a velocity of 10 metres a second is brought to rest in 1 second.

13. The resistance of the air being the same for two bodies of the same external form and substance; show that the velocity of a falling solid sphere will be less diminished in a given time than will be that of a hollow sphere of the same substance and external radius.

14. Taking the moon's distance as 240,000 miles, the earth's radius as 4000 miles, and a lunar month as $29\frac{1}{2}$ days; find approximately the periodic time of a satellite revolving close to the earth.

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15. If the units of acceleration and force be 4 sfas and 8 poundals; find the unit of mass.

16. Show that in equal sectors the arcs are inversely proportional to the radii.

17. Take two large French wire nails, and gently drive them so that they stand upright on a soft block of wood. Let their tops be in the same horizontal line. Now rest a heavy piece of iron (say a hammer-head) on one, and with another hammer give two equal blows, one on the free nail, the other on the hammer which is resting on the nail. It will be found that the free nail has gone the further into the wood. Explain this.

18. Take two small toy railway-trucks A and B, and place them on a smooth table. Make up the mass of A with coins to $2\frac{1}{2}$ oz., and let it be pulled by the force of a poundal. [This can be managed by tying $\frac{1}{2}$ oz. to the end of a string and letting it hang over the edge of the table.] Make up the mass of B to $1\frac{1}{4}$ oz., and let it be pulled by half a poundal. The trucks when let go travel nearly together, and when the poundal and $\frac{1}{2}$ poundal touch the ground the trucks are found at about the same place. Show that this is in accordance with the 2nd law.

19. At a place where g is 981, how long will a body, whose mass is 109 kilogrammes and velocity 9 metres a second, move against a force of 5000 g dynes?

20. It is generally assumed that forces can be represented by straight lines; show that this is a necessary consequence of the 2nd law.

21. Taking the periodic times of Ganymede and Callisto, round Jupiter, to be $(30)^3$ and $(39)^3$ seconds; find the ratio of their distances from the planet.

---

22. The units of mass, velocity and force being respectively a kilogramme, 49 cas, and 980 dynes; find the unit of time.

23. Would it be a true statement to say that the greater the force acting during a given time the greater the kinetic energy produced?

24. In question 18, when a coin is taken out of either truck, that one travels faster than the other; and is ahead when they come to rest. Explain this.

25. When a horse pulls a railway coach at a station, the first effort required is very great; but when the coach is once started very little effort is required to keep it going. Explain this more definitely than it is possible to do under the 1st law.

26. Find the retarding force, supposed constant, when a train of 73 tons, moving at the rate of 45 miles an hour, is brought to rest in $3\frac{1}{2}$ minutes.

27. The attraction of the earth for any body outside itself varying inversely as the square of the distance from the centre of the earth; and the polar and equatorial diameters of the earth being respectively 7899 and 7925 miles; if a mass at the pole have after falling 1″ a velocity of 32·255 fas; what will be approximately the velocity of a mass at the equator under the same circumstances?

28. If the distances of two planets from their primary be 425 and 675 thousand miles; and if the former revolve in 4 h. 5 m. 39 s.; find approximately the periodic time of the latter.

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29. During what time must a constant force equal to 32 British units act upon a cwt., to produce in it a velocity of a mile a minute?

30. If it be established that the greater the force acting over a given length, the greater the kinetic energy produced; what result follows?

31. Why does a mason, making a hole in a rock, use as small a chisel as is consistent with handiness and strength?

32. Take two ordinary kitchen "weights" of 2 lbs. and 4 lbs.; lay them on their sides on as smooth a table as possible,

attach thin strong strings to the rings, and pull them with forces respectively of 25 and 50 poundals. [The 25 and 50 poundals may be bags of sand, "weighing" $12\frac{1}{2}$ oz. and 25 oz., hanging over the edge of the table.] It will be found that the motions are as nearly as possible the same. Explain this.

33. A mass of 7 kilogrammes falls a certain height at a place where the acceleration of gravity is 980 scas; and then moves, with the velocity acquired, on a smooth horizontal plane. It is found that a force of $9\frac{4}{5}$ million dignes will now stop it in 3″; what height did it fall?

34. In the exhausted receiver of an air-pump a feather and a coin fall to the ground together. Why?

35. The distances of Enceladus and Dione from Saturn being about 160,000 and 250,000 miles; if the periodic time of the latter is 8000 seconds, what should be that of the former?

SECTION VII.

THE THIRD LAW OF MOTION.

on **138.** LAW 3.—**The Work Done by a force (or any agent) or any mass (or system of masses) has its equivalent in the Kinetic Energy exhibited, and in the Work Done against Molecular Forces, Gravity, and Friction.**

139. This law is, undoubtedly, more difficult to grasp than the two which have preceded it.

140. In general terms the statement seems reasonable enough.

When a canal-boat is towed the *work done* by the horse, it is reasonable to suppose, has its equivalent in the *energy* of the boat, and in the work done against the *friction* of the water.[1]

When a shot is fired into the air, then, speaking in general terms, the *work done* by the powder has its equivalent in the *energy* of the bullet, and in the work done against *gravity* and against the *friction* of the air.

When a spring is bent the *work done* has its equivalent mainly in the work done against the *molecular forces* of the steel.

When a mass is lifted the *work done* by the agent has its

[1] Probably work will be expended also against molecular forces and against gravity; but the amount will be small. And so in other cases.

equivalent in the *energy* of the mass, and in the work done against *gravity.*

In a "Tug of War" the difference between the *work done* by the two sides, it is reasonable to think, is equal to the *kinetic energy* exhibited.

141. All this seems reasonable enough, provided that we are only speaking in general terms.

It is when we come to give a definite meaning to Work Done that difficulty arises.

142. DEFINITION.—In cases of motion under the action of a force, the point of application of the force moves through a certain length in the direction of the force; and we say usually that the *product* of the number of units of *force* by the number of units of *length* moved, is the number of units of *Work Done* by the force.

The *definition* of Kinetic Energy, as given in Section IV., is: the *Kinetic Energy* of a body is that quality the number of units of which is equal to *half* the *product* of the number of units of *mass* by the *square* of the number of units of *velocity.*

143. If p units of force
have moved a mass of m „ mass
through x „ length in the dir. of the force
and given it v „ velocity;
then, the number of units of work done is px,
and the number of units of K. E. exhibited is $\frac{1}{2}mv^2$;
and the 3rd law asserts that

$$px = \tfrac{1}{2}mv^2 + \left\{ \begin{array}{l} \text{the work done against molecular} \\ \text{forces, gravity, and friction.} \end{array} \right\}$$

144. All we can hope to do here towards establishing the truth of this equation is to acknowledge that, in general terms, the law seems reasonable, to inquire what mathematical form the law takes in a simple case, and to allow this to guide us to the mathematical form of the more general statement.

145. Now in Section IV. we arrived at this statement,

$$ma \times x = \tfrac{1}{2}mv^2;$$

where *ma* sfasp is the mass-acceleration
owing to which, in *x* feet,
the mass of *m* pounds acquires
a velocity of *v* fas.

Therefore, by the 2nd law, *ma* is the number of units of force acting on the mass; and this equation, in consequence, represents the 3rd law for a body moving from rest under the action of a constant force; when no molecular force, gravity, or friction is taken into account.

It seems reasonable to suppose that a similar equation holds in the more general case; which equation is the mathematical expression of the 3rd law.

146. A term which it is necessary now to explain is Potential Energy.

Kinetic Energy is exhibited by the actual velocity developed. Potential Energy is not exhibited by velocity, but is yet a most important result of the Doing of Work.

147. When part of the work is done against *Molecular Forces*, as in the 3rd illustration above, the recoil of the spring is capable at some future time of reproducing the work (or part of the work) originally done.

So that in the bent spring there is *Energy stored up.*

So that we may put the illustration thus:

When a spring is bent the *work done* by the agent has its equivalent mainly in the *Energy stored up* in the spring.

The adjective "*Potential,*" which means "*existing in possibility, not in act,*" is used to designate this sort of energy; and therefore

When a spring is bent the *work done* by the agent has its equivalent mainly in the *Potential Energy* in the spring.

Thus, again, when a bow is bent the *work done* has its equivalent mainly in the *Potential Energy* laid up ready to be converted into the Kinetic Energy[1] of the arrow.

148. Similarly with work done against *Gravity.*

149. In Newton's day and long afterwards it was supposed that work done against *Friction* was absolutely lost.

As a fact it has its equivalent in *heat*; though it cannot be stored up for future use as in the work (or part of the work) done against molecular forces and against gravity.

In every case in which work is "lost" by friction heat is generated; and Joule's investigations have shown that the

[1] Called "Kinetic," to distinguish from "Potential" (from the Greek word κινητικός).

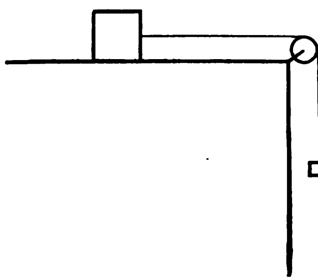
quantity of heat so generated is a perfectly definite equivalent for the work "lost."

This Friction includes various resistances, as, *e.g.*, those arising from Friction between solids, Viscosity of fluids, Imperfect elasticity of solids, Induction of electric currents, etc.

150. Work therefore is never lost, though it may be dissipated like a spendthrift's gold; or be useless and irrecoverable like the labour of Sisyphus.

151. A very important result of the 3rd law has yet to be mentioned.

If M lbs. on an ice-table be drawn by m lbs. hanging over the edge; and if the whole length of the string be a feet; and the part hanging, when the velocity is v fas, be x feet (m being, at starting, on a level with the top of the table);



and if m' lbs. be the mass of the string;

also if T poundals be the action of M on the string;

and T' poundals the action of m on the string;

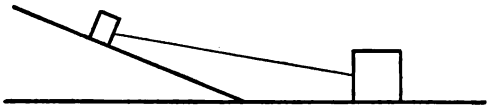
then, for the motion of the *string*, neglecting its weight,

$$\frac{1}{2}m'v^2 = T'x - Tx.$$

Now if we may neglect the mass[1] of the string as being very small compared with M and m pounds, we get the remarkable result

$$T' = T.$$

152. Similarly in such a case as that here represented, where a mass of m lbs. falling down an incline pushes M lbs., lying on a horizontal plane, by means of a thin rod,



whose mass may be neglected; the pressure of m on the rod is equal throughout the motion to the pressure of M on the rod.

153. Massless strings and rods do not actually occur in nature; but there is a great principle involved in these results.

For when a pull or a push occurs directly, *i.e.* without the intervention of a string or rod; then throughout the motion the pressure each way is the same.

154. Thus, in the 1st example, M pulls the string with the same force as that with which the string pulls M.

And so *their mutual action has no effect on the motion of the whole system.*

[1] We may not always neglect this mass. Thus suppose a watch to be lying on a polished table with a portion of the chain hanging over the edge. The watch will be drawn off the table; and, in working at such a problem, the mass of the whole chain and the weight of the portion hanging would certainly have to be considered.

155. It is this which is the important principle, and which in general terms may be thus stated :

isn't this implied in the idea of rigidity?

The Internal Pressures of any system of rigid bodies are in Equilibrium amongst themselves.

WORKED EXAMPLES.

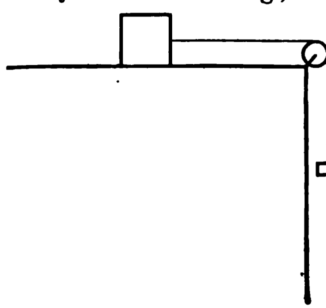
A. If a carriage is being drawn up an incline on which the acceleration of gravity has but 1-70th of the effect that it has on a body falling freely ; and if the rope break when the carriage has a velocity of 46 fas ; how far, neglecting friction, will the carriage continue to move up the incline ? [$g=32{\cdot}2$.]

When it stops the kinetic energy is just all used up ; so, if m lbs. be the mass of the carriage and x ft. the distance it goes,

$$m\frac{32{\cdot}2}{70}x=\frac{1}{2}m(46)^2\,;$$

whence, the required distance is 2300 feet.

B. If M lbs. on an ice-table is drawn by m lbs. hanging over the edge, the two being connected by a massless string ; find the tension of the string.



As the string is massless no force is required to move it ; so that the tensions at either end are equal.

For the motion of M, we have,

$$Tx=\frac{1}{2}Mv^2\,;$$

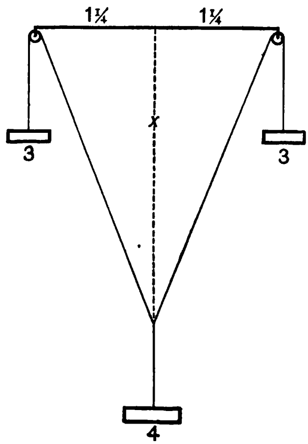
and for the motion of m

$$(mg - T)x = \frac{1}{2}mv^2\,;$$

$$\text{whence } T = \frac{Mmg}{M+m}.$$

Shorter from the equality of accel. in the two branches of the string: $a = \frac{T}{M} = \frac{mg-T}{m}$ $\therefore T =$

C. Two masses of 3 lbs. each are tied at the ends of a massless string which passes over two smooth massless pulleys in the same horizontal line, 30 inches apart. A mass of 4 lbs. being fixed at the point of the string midway between the pulleys; find how far it will descend.



Let x feet be the required distance. At the lowest point the kinetic energy vanishes; and therefore the total work done is zero.

Now the 3 lb. masses have each ascended through

$$\left\{\sqrt{x^2+\frac{25}{16}}-\frac{5}{4}\right\}\text{feet}\,;$$

$$\therefore\ 4\times x-3\times\left\{\sqrt{x^2+\frac{25}{16}}-\frac{5}{4}\right\}-3\times\left\{\sqrt{x^2+\frac{25}{16}}-\frac{5}{4}\right\}=0\,;$$

whence the distance $=3$ feet.

D. At a place where the acceleration of gravity is 981 scas, a stone is whirled round in the circumference of a vertical circle whose radius is 30 centimetres; if the velocity at the highest point is 161 cas; what is it at the lowest point?

The velocity at the lowest point is greater than that at the highest point by the work done in the interval by gravity; and as the distance through which the stone has fallen is the diameter of the circle,

$$\therefore \frac{1}{2}mv^2 = \frac{1}{2}m(161)^2 + mg \times 60;$$

$$\therefore v^2 = (161)^2 + 2 \times 981 \times 60;$$

whence the velocity is 379 cas.

E. If the Action of an agent be measured by its velocity and amount conjointly; and if the Reaction of the resistance be measured by the velocities of the several parts and their amounts, arising from molecular forces, gravity, friction, and *acceleration*; what connection is there between the Action and the Reaction?

Equal and Opposite; for Action, as defined here, is clearly proportional to Work Done; and Reaction to the Work Done against molecular forces, gravity and friction, together with the *kinetic energy* developed.

This is Newton's full statement of the 3rd law.

It is also "D'Alembert's principle."

F. Supposing a mass to be projected vertically upwards. It starts with an amount of kinetic energy. At a certain height the whole of that kinetic energy will have disappeared; so that one might take the mass into one's hand without exertion. Explain the position.

Part of the kinetic energy has been expended in work against gravity, and part in work against the friction of the air.

The work done against gravity is now stored up as Potential

Energy, the greater part[1] of which could be again turned into Kinetic Energy by allowing the mass to fall.

So we may say that if a mass be projected vertically upwards; then at a certain height the whole of the kinetic energy with which it started will have disappeared, but will have its equivalent in potential energy and in the work done against the friction of the air.

EXAMPLES.—VII.

1. A stone is thrown vertically upwards, at a place where the acceleration of gravity is 32·12 sfas, with a velocity of 219 miles an hour; how high will it go?

2. At a place where the acceleration of gravity is 981 scas, two bodies of mass 1 kilogramme are connected by a massless string which passes over a massless and frictionless pulley at the top of a smooth inclined plane, such that the earth's attraction has only $\frac{1}{3}$rd of its power on masses lying on it. Find the scasgram producible by the tension of the string.

3. If a hand on which is resting a mass of m lbs. move vertically downwards with an acceleration of a sfas at a place where the acceleration of gravity is g sfas; find the pressure of the mass on the hand.

4. In the illustration to the 1st law, in which a mass is projected horizontally, show (by the 3rd law) that the velocity at any point of the parabola is the same as if the mass had *fallen* from the directrix.

5. If a marble falls on a pavement and rebounds; show that it loses energy by the impact.

[1] Some is "lost" owing to the air-friction.

6. When a mass slides from rest down a smooth inclined plane under the action of gravity only, show that (neglecting air-friction) the total energy is constant.

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7. A mass of 100 kilogrammes is moving with a velocity of 7 metres a second; of how many dynes is the force which will stop it in 25 metres?

8. At a place where g is 980, a balloon is ascending, and a mass of 500 grammes presses on the floor of the balloon with a a force of 550 g dynes; find the acceleration; and the velocity at a height of 100 metres.

9. A hand is holding a mass of 5 lbs. at a place where the acceleration of gravity is 32·2 sfas; if the hand move downwards with an acceleration of 25 sfas, find the pressure between the mass and hand.

10. A mass of 770 lbs. leaves a gun of $36\frac{1}{2}$ tons with a muzzle velocity of 1168 fas; the recoil is checked by a constant force of 18,480 g poundals ($g = 32{\cdot}12$); in what space will the gun's velocity be reduced one-half?

11. Two equal volumes of the same substance fall in the same time whether dropped separately or joined together. What does this indicate concerning the mutual pressures, when they are joined together?

12. When a bow is bent, what becomes of the work done?

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13. What mass, moving with a velocity of a quarter-mile a minute, can be stopped in 22 yards by a force which, at a place where g is 32·12, is equal to 25 g poundals?

14. If a mass slide down a smooth inclined plane, it is found that (neglecting air-resistance) the velocity attained by falling down the slope of the plane is the same as that attained by falling down the vertical height. Show that this corroborates the 3rd law.

15. Two masses of m lbs. each are attached to the ends of a massless string, which passes over two smooth massless pulleys in the same horizontal straight line 3 feet apart. Find that mass which, being fixed at the point of the string midway between the pulleys, will just descend 2 feet. Cf. C p. 93

16. At a place where the acceleration of gravity is 980 scas, a stone is whirled round in the circumference of a vertical circle whose radius is 1 metre; if the velocity at the lowest point is $6\frac{9}{20}$ metres per second; what is it at the highest point?

17. A limited system of bodies being called "*Dynamically Conservative*" when the mutual forces between the parts always perform the same amount of work during any motion between the same two positions; show that in nature there is *apparently* no such system. What must be taken into account that a system may be regarded as conservative?

18. If a particle move in any manner, under the action of gravity only, show that the potential energy lost is equal to the kinetic energy gained.

---

19. The last carriage breaks away while a train is going up an incline on which the acceleration of gravity has but 1-70th of the effect that it has on a body falling freely. The carriage continues to move up the incline for 700 metres; find (neglecting friction) the velocity with which the train was moving. [$g=980$.]

20. At a place where the acceleration of gravity is 32·2 sfas

two masses of 7 lbs. and 13 lbs. 2 oz. hang at the ends of a massless string which passes over a smooth massless pulley; find the velocity after passing over 10 feet from rest.

21. If a man jump off a bank with a mass on his head; find the pressure on his head during the fall.

22. At a place where the acceleration of gravity is 981 scas, find the velocity of a projectile (in vacuo) after it has risen 38 centimetres above the point of projection; the initial velocity being 1000 cas.

23. If two marbles collide, show that there must be loss of energy.

24. Frame the law which shall bear the same relation to the 3rd law, that the 1st law does to the 2nd law.

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25. A hammer of mass 1 kilogramme just comes to rest after driving a nail 1 centimetre; if the velocity of the hammer at the moment of striking be 10 metres per second, find the total force (supposed constant) exerted against the hammer; and the duration of the action.

26. Show that the number of fasp produced by a constant force in t'' is to the number of faspen produced in x feet, as $t : x$.

27. If a mass be suspended from a spring-balance, and the whole be moved upwards with an acceleration equal to that of gravity; what will be the result?

28. To how many dynes is the force equal, which must act on a mass of $24\frac{1}{2}$ kilogrammes to increase its velocity from 1000 cas to 1400 cas, while it passes over 20 metres?

29. In a "Tug of War," in whose favour is the mass of the rope?

30. Show that (neglecting air-resistance) the energy of a projectile is constant.

---

31. A train running at 21·9 miles per hour just comes to rest under the action of friction in $4\frac{2}{3}$ miles. Taking the force of friction to be constant, find in poundals the least force per ton which would make the train move from rest.

32. In qu. 2 above, if the *mass* of the string be 1 gramme, find in poundals the tensions at either end; neglecting the *weight* of the string.

33. If a mass revolve in a circle round a centre of force in the centre of the circle; show that the velocity is constant.

34. If unit mass revolve round a centre of force, find the relation between the acceleration to the centre and the kinetic energy gained for a small increase of the radius vector.

35. Show that the action of the Moon on the Earth's ocean tends to make the earth turn always the same portion of her surface towards the moon.

36. A mass of m lbs. is projected vertically upwards, at a place where the acceleration of gravity is g sfas, with a velocity of V fas. What, in faspen, are its Kinetic and Potential Energies after it has been moving for t''?

SECTION VIII.

FORCE, WEIGHT, IMPULSE, WORK, POWER.

156. By the 2nd law of motion we see that

That which produces Momentum is **Force**;

The corresponding change of momentum is proportional to the force;

Force is measured by the momentum produced in a given time.

157. We may define the Unit Force[1] in any one of the following ways:

A. Unit Force is that which in unit time produces unit momentum;

or (separating the mass from the momentum)

B. Unit Force is that which in unit mass in unit time produces unit velocity;

or (now combining the time and the velocity)

C. Unit Force is that which in unit mass produces unit acceleration;

or (more briefly)

D. Unit Force is that which produces unit mass-acceleration;

or (resolving the velocity in **B**)

---

[1] Called *Gauss' Absolute Unit*;—absolute because it gives a standard independent of gravity.

E. Unit Force is that which produces in unit mass in unit time a velocity of unit length per unit time.

158. A, B, C, D and **E** may, of course, be expressed more definitely in terms of either British or *C.G.S.* units.

Thus the British Unit Force is that which in 1 pound in 1 second produces 1 fas.

Or, again, the *C.G.S.* Unit Force is that which in 1 gramme produces 1 scas.

159. Weight is but a particular kind of force.

A force is that which (see **B** above) produces in a certain mass in a certain time so much velocity.

When the force which does this is the earth's attraction, the force is called *Weight*.

160. As force and weight are things of the same kind, it is possible and most useful to express force in terms of the weight of some mass.

161. Now, at a place where the acceleration of gravity is 32 sfas,

The weight of	1 lb.	is that which	produces in	1	lb. mass	32	sfas;
∴ „	$\frac{1}{32}$ lb.	„	„	1	„	1	„
But the	British Unit of Force	„	„	1	„	1	„

Therefore the British Unit of Force is at such a place equivalent to the weight of a mass of $\frac{1}{32}$ lb., or of half an ounce.

162. At no place *on the surface* of the earth is the acceleration of gravity so little as 32 sfas.

At a place where the acceleration of gravity is g sfas, the British Unit Force is equivalent to the weight of a mass of $\frac{1}{g}$ lb.

As g, at the surface of the earth, is greater than 32·088 and less than 32·255, this unit is the weight *on the surface of the earth* of a mass (varying within small limits according to the latitude, but always just) a little under half an ounce.[1]

DEFINITION.—This Unit of Force is called a *Poundal.*

163. Notice again carefully that it is only at or near the *surface* of the earth that the weight of this half-ounce mass can be taken to represent the Poundal.

The Poundal is the British unit of force (*i.e.* is the unit of force expressed in terms of British units), and is invariable throughout the Universe. The half-ounce mass is also everywhere an invariable thing. But it so happens that, at the surface of the earth, the *weight* of this particular mass is very nearly equal to a Poundal, and, by remembering this, we get a good conception of what a Poundal is.

164. DEFINITION.—The *C.G.S.* unit of force is called a *Dyne.*

A Dyne is that which in 1 gramme mass, in 1″, produces 1 cas.

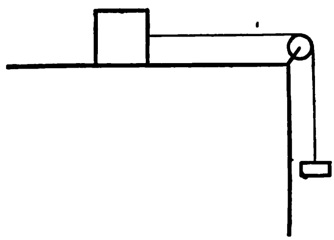
[1] A "mass of one ounce" is, of course, one-sixteenth of the mass of the London platinum pound.

165. Referring to paragraph **60**, it is seen that a Dyne may be measured by the weight, at the *surface* of the earth, of a mass a little greater than the one-thousandth part of a gramme. Weight of $\frac{1}{981}$ (= about ·001) = 1 dyne

166. ILLUSTRATION.—A poundal is that which

in a mass of 1 lb., in 1″, produces 1 fas.

Place a mass of $15\frac{1}{2}$ oz. on a perfectly smooth horizontal table, in a place where $g = 32$; and, by a weightless string, attach to it a half-ounce mass; and let this mass hang over the edge of the table. Then, since the total mass acted on is ($15\frac{1}{2}$ oz. + $\frac{1}{2}$ oz. =) 1 lb., and the force is a poundal, (neglecting the resistance of the air, and reckoning from the instant when the mass is let go), the mass will in 1″ have acquired a velocity of 1 fas.



On the surface of the earth, where g is always greater than 32, the hanging mass must be less than $\frac{1}{2}$ oz. to produce the same result.

ap. 180 At a point about half-way between the centre and the surface of the earth, a mass of an ounce would be required.

At the centre of the earth this method of producing a sfas in a 1 lb. mass would be impossible. As there is no gravity, some other method of producing force would have to be employed.

167. It may be usefully noted that a Poundal and a Sfasp are equivalent things looked at from different points of view.

A poundal is that which produces a sfasp.

So a dyne is that which produces a scasgram.

It is convenient to retain both terms, just as it is convenient to retain in money denominations both 1 sovereign and 20 shillings.

168. Impulsive Force.—If a force has the nature of a blow, it is convenient sometimes to have a measure of it differing somewhat from that above given.

169. A poundal is that which acting for 1″ produces 1 fasp.

If the poundal ceases to act at the end of the 1″, we may say that the poundal has produced 1 fasp.

If a force which only acted for $\frac{1}{2}''$ were required to produce 1 fasp, the force must be 2 poundals.

And so n poundals acting for $\frac{1''}{n}$ produce 1 fasp.

170. Now let n be large, then we have a large force acting for a short time and producing 1 fasp.

Such a force is of the nature of a blow, and is usually called an *Impulsive Force*; and its effect is called an *Impulse*.

171. Impulsive forces are common; they occur, *e.g.* in the firing of a gun, in any explosion, in the hitting of a ball with a bat, in the driving of a nail, etc.

172. As we cannot, in general, estimate the time during which an impulsive force acts, we cannot know the force properly; but we may compare two such forces by the numbers of fasp or casgram produced.

173. The *Unit of Impulse* is the impulse of a force which generates *Unit Momentum.*

The British Unit of Impulse is, therefore, the impulse of a force which generates a Fasp.[1]

The *C.G.S.* Unit of Impulse is the impulse of a force which generates a Casgram.[2]

174. Work.—When (as in very many practical examples) the kinetic energy is the same at the beginning and end of the motion, then (neglecting molecular forces) the *Work Done by an Agent* is simply equal to the Work Done against Gravity and Friction.

When, in such cases, we know the distance moved and the time taken, we are said to know the *Rate at which the Agent Works*, and its *Power.*

175. Just as Weight gives a convenient measure of Force, so *Work Done against Weight*[3] gives a convenient measure of Work.

[1] This we may, for brevity, call a **Blm.**

[2] This we may call a **Clm.**

[3] It is worth while noting how well a sentence of this kind brings out the distinction between Mass and Weight. Mass is a quality of the body itself; but Weight is an outside influence, an external force which acts upon the body.

176. Suppose a mass of half-an-ounce to be raised vertically by an agent, at a place where the acceleration of gravity is 32 sfas, the kinetic energy being the same at the beginning and end of the motion.

Then a *British Unit of Work* will be done by the agent if the half-ounce mass be raised through a foot; and this unit of work is, for obvious reasons, called a *Foot-Poundal.*

177. In general,

A unit of work is done if $\frac{1}{g}$ (unit mass) be lifted through unit length;

or, more generally, since the weight of $\frac{1}{g}$ (unit mass) is the unit force,

A *Unit of work* is done if the point of application of unit force is moved by any agent through unit length in the line of action of the force, but in the contrary direction to that in which the force acts.

178. Definition.—When the point of application of a Poundal is moved through a foot in the direction of the Poundal, we have the *Foot-Poundal,* the British unit of work.

Definition.—When the point of application of a Dyne is moved through a centimetre in the direction of the Dyne, we have the *Erg,* the *C.G.S.* unit of work.

179. Definition.—An engineer's *Foot-Pound* is the work done in raising a mass of *one pound* through one foot.

180. It is clear that the amount of work represented by a Foot-Pound is variable.

For the weight of a pound being g poundals, a foot-pound is g foot-poundals.

At the centre of the earth (where g is nothing) a foot-pound would also be nothing.

Half-way from the centre of the earth to the surface it would have but half the value that it has at the surface.

And on the surface of the earth itself the value of the Foot-Pound varies according to the latitude.

181. Nevertheless the variation *over the surface of the earth* is comparatively small; and the value of the Foot-Pound is sufficiently constant and exact for many practical purposes; though it is unsuited for a *scientific* unit of work.

182. The clumsiness of the Foot-Pound system is in great contrast to the clear and simple accuracy of that of the Foot-Poundal.

183. The **Power** of any agent depends, as we have seen (Paragraph **174**), not only on the number of foot-pounds or of foot-poundals it can perform; but also on the time taken for the performance; *i.e.* on the rate at which the agent works.

184. Definition.—The agent which can perform 33,000 foot-pounds in a *minute* is said to be *Of one Horse-Power.* or 550 ft-lb per sec.

This practical unit of a Horse-Power (introduced by Watt) will, of course, vary from place to place according to the

value of g; and would not be a right measure of work at places much removed from the surface of the earth.

A more exact statement would be that a Horse-Power is the power of an agent which can perform per *minute* rather more than a million[1] foot-poundals.

185. The British Unit of Power is that of an agent which can perform one foot-poundal per second.

In like manner the *C.G.S.* Unit of Power is the power of doing work at the rate of one erg per second.

186. At a place where

$$1 \text{ H.P. performs } 1062000 \text{ foot-poundals in } 1 \text{ minute};$$
$$60 \quad ,, \quad 1062000 \quad ,, \quad 1 \text{ second};$$
$$\therefore \frac{60}{1062000} \quad ,, \quad 1 \quad ,, \quad 1 \text{ second};$$
$$\therefore \frac{60}{1062000} \text{ H.P.} = \text{the unit of power};$$

$\therefore$ the British Unit of Power is $\frac{1}{17700}$ of the H.P. at that place.

187. On the surface of the earth the Horse-Power contains from 17,550 to 17,750 British units of power.

188. Notice again carefully that *Mass, Density, Poundal, Foot-Poundal,* etc., are things which never vary; whereas *Weight,* and all units that depend on Weight, vary with the locality. Such units are called Gravitation Units.

[1] From about 1,059,000 to about 1,064,000.

WORKED EXAMPLES.

A. If a force equal to the weight of 1 cwt. acts for the 28th part of a minute on a mass of 1 ton; find the number of fasp acquired at a place where the acceleration of gravity is 32·1 sfas. i.e. momentum

The force

is that which acting on 112 lbs. for 1″ prod. a vel. of 32·1 fas;

$$i.e. \quad ,, \quad ,, \quad \underbrace{20 \times 112 \text{ lbs.}}_{= 1 \text{ ton}} \quad ,, \quad \frac{60''}{28} \quad ,, \quad \frac{32·1}{20} \cdot \frac{60}{28} \quad ,,$$

$= 112 \times 32·1 \times \frac{60}{28}$

$$\therefore \text{ no. of fasp} = 20 \times 112 \times \frac{32·1 \times 60}{20 \times 28} = 7704.$$

Note that this result would be the same whatever the mass acted on.

B. If the number of units of weight in a body be 1536 times the number of units of mass; and if the unit of length be an inch; find the unit of time; when the acceleration of gravity is 32 sfas.

If t'' and m lbs. be the units of time and mass; and if M lbs. be the mass of the body; the weight of the body is that which produces in M lbs. in 1″ a vel. of 32 ft. per 1″;

$$i.e. \quad ,, \quad ,, \quad m \text{ lbs. in } t'' \quad ,, \quad \frac{32M \times t^2 \times 12}{m} \text{ ins. per } t'';$$

$$\therefore \frac{32M \times t^2 \times 12}{m} = 1536 \times \frac{M}{m};$$

$$\therefore t^2 = \frac{1536}{32 \times 12} = 4;$$

and the unit of time is 2″.

C. How many Bim will be required to project 1 lb. vertically 1 foot at a place where the acceleration of gravity is 32 sfas?

If m lbs., v fas and x feet be the mass of a body, the velocity of its projection upwards, and the height to which it rises; then

$$\frac{1}{2}mv^2 = 32m \times x,$$

since the full work done is equal to the original kinetic energy;

$\therefore$ with the data of the problem

$$v = \sqrt{64} = 8;$$

$$\therefore \ mv = 8;$$

$\therefore$ the number of Bim is 8.

---

i.e. impulse

D. If a shot of half a ton leave a fixed gun with a velocity of 1000 fas; find the number of Bim that have acted on the shot; and the time taken, supposing the gun 10 feet long, and the acceleration constant.

No. of Bim is, of course,

$$10 \times 112 \times 1000 = 1120000.$$

Since $$\frac{1}{2}mv^2 = ma \times x,$$

$$\therefore \ a = \frac{v^2}{2x} = \frac{10^6}{20};$$

and since $$mv = ma \times t,$$

$$\therefore \ t = \frac{v}{a} = \frac{10^3 \times 20}{10^6};$$

and the time is $\frac{1}{50}$ of a second.

simplest method:

$\frac{1}{2}v^2 = as, \therefore a = \frac{v^2}{2s}$

$v = at, \therefore t = \frac{v}{a}$

$\therefore t = \frac{2s}{v} = \frac{2 \times 10}{1000}$

E. An engine is drawing a train, of mass 85,000 kilogrammes, at $84\frac{4}{5}$ kilometres an hour, consuming 900 kilogrammes of coal in the hour. The resistance to motion being equal to the

weight of $4\frac{1}{2}$ grammes per kilogramme of mass; 1 gramme of coal being capable of raising 80 grammes of water from freezing to boiling; and the raising of 1 kilogramme of water through 1° centigrade being equivalent to 42,400,000 ergs; find what proportion of the heat generated is usefully employed.

The ergs done by the engine in 1″ are

$$85000 \times 4\tfrac{1}{2} \times \frac{84\frac{4}{5} \times 10^3 \times 10^2}{60 \times 60};$$

?

while the ergs that should arise in 1″ from the coal consumed are

$$900 \times 1000 \times 80 \times \frac{100 \times 42400000}{1000} \times \frac{1}{60 \times 60};$$

the required proportion is the ratio of these two, or $\frac{17}{1600}$.

F. If an agent perform a unit of work in a unit of time, when the units of velocity, acceleration and mass are 50 fas, the acceleration of gravity and 110 lbs.; of how many Horse-Power is the agent (there being no change of K. E.)?

Unit force is, clearly, the weight of 110 lbs.;

Unit acceleration is

that which shows in t'' a velocity of $50t$ feet per t'';

$\therefore$ unit length is $50t$ feet;

$\therefore$ the unit of work $= 110 \times 50t$ foot-pounds;

$$\therefore \text{ the H.P. of the agent} = \frac{110 \times 50t \text{ foot-pounds in } t''}{550t \text{ foot-pounds in } t''}$$

$$= 10.$$

G. A carriage which requires a force equal to the weight of 50 lbs. is drawn over 10 miles in 1 hour 20 min.; find the

number of foot-poundals done at a place where the acceleration of gravity is 32 sfas; and compare the rate of working with a horse-power.

$$\text{No. of ft.-poundals} = 50 \times 32 \times 10 \times 1760 \times 3;$$
$$= 84480000.$$

The animal does $\frac{50 \times 10 \times 1760 \times 3}{80}$ foot-pounds in 1′;

1 H.P. is equivalent to 33,000 foot-pounds in 1′;

$\therefore$ required comparison $= \frac{50 \times 10 \times 1760 \times 3}{80 \times 33000} = 1$;

$\therefore$ rate of working $= 1$ H. P.

EXAMPLES.—VIII.

+ 1. During what time must a force, equal to the weight of 1 lb., act upon 1 cwt. to produce in it a velocity of a mile a minute? [The acceleration of gravity is 32 sfas.]

2. If the units of force and acceleration be respectively the weight of 4 oz. and 4 sfas; find the unit of mass at a place where the acceleration of gravity is 32 sfas.

3. If 1000 yards be taken as equivalent to 915 metres, and 1 lb. to 454 grammes; how many Cim are there in a Bim?

+ 4. If a shot of 2 oz. in one gun rise to a height of 225 ft., and a shot of 3 oz. in another to a height of 144 ft.; compare the masses of powder used, assuming that the shot-momentum caused by the explosion is proportional to the mass of the powder.

5. At a place where the acceleration of gravity is 981 scas, how many ergs are done against gravity in lifting (i) 1 gramme through 1 centimetre; (ii) 1 kilogramme through 1 metre?

6. If the units of mass, force and work be respectively 4 tons, 1 ton and 449,680 foot-pounds; find the units of time and length at a place where the acceleration of gravity is 32·12 sfas.

7. The mass of a train being 100 tons, and the resistance 10 lbs. weight per ton; find the H. P. sufficient to keep the train going at the rate of 30 miles per hour.

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8. A mass of 10 lbs. 1 oz. depresses the marker of a spring-balance, at a place where $g=32{\cdot}1$, to a point P; what mass will depress the marker to P, at a place where $g=32{\cdot}2$?

9. At a place where the acceleration of gravity is 980 scas, the units of length and force being respectively 7 centimetres and the weight of 4·9 grammes, and the mass of a cubic centimetre of unit-density substance being 2 grammes, find the unit of time.

10. How many Cim will be required to project 1 gramme vertically 1 centimetre, at a place where the acceleration of gravity is 980 scas? [Take $\sqrt{10}=3\frac{5}{28}$.]

11. A marble of a certain material when dropped on a pavement rebounds with $\frac{3}{7}$ of the velocity with which it reaches the pavement. If such a marble, massing 50 grammes, be dropped from a height of $2\frac{1}{2}$ metres, at a place where the acceleration of gravity is 980 scas; find the number of Cim developed.

12. Taking a pound as $\frac{9}{20}$ of a kilogramme, and a foot as $30\frac{1}{2}$ centimetres; how many ergs does this give to the foot-pound at a place where the acceleration of gravity is 32 sfas; and how many to the foot-poundal?

13. If 1 pound, 32·12 sfas, and 73 H. P. be respectively the

units of mass, acceleration, and power; find the units of time and length.

14. Find the H. P. required to move a steamer at the rate of 1000 feet per minute; the resistance to the steamer's motion being taken as equal to the weight of 33 tons.

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15. The pull on a train being the weight of a ton, the mass of the train 100 tons, and the resistance 10 lbs. weight per ton; find the acceleration, the acceleration of gravity being 32 sfas.

16. At a place where the acceleration of gravity is 980 scas, what number of weights of 1 gramme is the unit of force, when the units of mass, length, and time are respectively 21 kilogrammes, 21 metres, and a minute?

17. How many Cim will be required to project 1 kilogramme 10 metres high, at a place where the acceleration of gravity is 980 scas?

18. If a shot of 100 grammes in one gun rise to a height of 64 metres, and a shot of 200 grammes in another to a height of 49 metres; compare the masses of powder used, assuming that the shot-momentum caused by the explosion is proportional to the mass of the powder.

19. If a horse, pulling a cart at the rate of a metre a second (at a place where the acceleration of gravity is 980 scas), is working at the rate of 6860 millions *c.g.s.* units of power; find the force of the horse's pull in kilogrammes.

20. If the units of acceleration and density be respectively that of gravity and that of water; and if 896 units of work be required to raise $\frac{3}{4}$ of a ton through 2700 feet; find the unit of length; a cubic foot of water massing 1000 oz.

21. The mass of a train being 110 tons, and the resistance 10 lbs. weight per ton; find the speed with which it can be drawn by an engine of 80 H. P.

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22. In question **15**, after what interval from rest will the train have a velocity of 31 fas?

23. Taking a gramme as ·002205 lb., and a centimetre as ·0328 ft.; how many dynes approximately does this give to the poundal?

24. How many Bim will be required to project 5 lbs. vertically upwards 16·1 feet, at a place where the acceleration of gravity is 32·2 sfas?

25. If two railway trucks rebound with $\frac{4}{5}$ of the velocity with which they strike; find the number of cim which act on each of two trucks (of mass 4000 kilogrammes) which collide, when each is moving with a velocity of $\frac{1}{4}$ kilometre per minute.

26. Taking a kilogramme as $2\frac{1}{5}$ lbs., and a centimetre as $\frac{2}{5}$ inch; what decimal of a foot-pound does this make the erg, at a place where the acceleration of gravity is 32 sfas; and what decimal of a foot-poundal?

27. If the number of units of work done in lifting a cubic foot of water (mass 1000 oz.) through 1 foot be 12 times the number of units of weight in the cubic foot, 18,432 times the number of units of mass in it, and 1000 times the number of units of volume in it; find the units of length and mass.

28. The mass of a train being 100,000 kilogrammes, and the resistance 5 grammes per kilogramme; find the speed with which it can be drawn by an engine capable of doing $600g$ million ergs per second; (g scas being the acc. of gravity.)

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29. Find the acceleration with which a lift is descending, at a place where $g=32{\cdot}2$; when the pressure of a man of 10 stone on the floor of the lift is 120 lbs.

30. If the acceleration of gravity be 979·69 scas, and the units of length and force be a centimetre and the weight of the unit mass; what is the unit of time?

31. At a place where the acceleration of gravity is 980 scas, two masses m_1, m_2, are projected vertically upwards, there being 7 times the number of Cim impressed on m_2 that there are on m_1. m_1 ascends 10 metres; m_2 is in the air altogether 10 seconds. Find the ratio $m_1 : m_2$.

32. A mass of 1 lb. of putty falls from a height of 12 feet on one scale of a balance, at the same moment that two other masses of putty fall on the other scale from heights of 27 feet and 100 inches. Find the two masses that the balance may remain permanently at rest.

33. An engine is drawing a train of 86 tons mass, at 54 miles an hour, consuming $\frac{55}{56}$ of a ton of coals in the hour. The resistance to motion being equal to the weight of 10 lbs. per ton of mass; 1 lb. of coal being capable of raising 80 lbs. of water from freezing to boiling; and the raising of 1 lb. of water through 1° Fahrenheit being equivalent to 774 foot-pounds; find what percentage of the heat generated is usefully employed.

34. If the units of length, mass, and work be respectively 1 mile, $766{,}656 \times 10^6$ lbs., and 1 ton raised $\frac{110}{7}$ foot; find the unit of time; the accn of gravity being 32 sfas.

35. At a place where the acceleration of gravity is 980 scas, how many ergs per second are equivalent to 1 H. P.; a pound being taken as equivalent to 453 grammes, and a foot to $30\frac{1}{2}$ centimetres?

36. At a place where the acceleration of gravity is 32 sfas, the units of time, density, and force are respectively $\frac{1}{50}$ second, half the density of water, and the weight of a cubic foot of water; find the units of length and mass (a cubic foot of water massing 1000 oz.).

37. If the unit of length be a metre, find approximately the unit of time, that the numbers of units of weight and mass (in the same body) may be the same. [$g=979{\cdot}69$; with C.G.S. units.]

38. At a place where the acceleration of gravity is 32·12 sfas, two masses of m_1 and m_2 lbs. are projected vertically upwards; there being 5 times the number of Bim impressed on m_2 that there are on m_1. m_1 is 8 seconds in the air, m_2 rises $401\frac{1}{2}$ feet. Find the ratio $m_1 : m_2$.

39. Three masses of putty fall, from heights of 1, 4 and 9 metres, on to the scales of a balance at the same moment. If the whole mass of putty is a kilogramme, find the masses of the three pieces that the balance may remain permanently at rest.

40. At a place where the acceleration of gravity is 980 scas; find the work done in lifting 1000 kilogrammes through a quarter of a kilometre; when the mass has at the end a velocity of 10 metres a second.

41. If the units of time, length and mass be respectively 805″, 16,100 feet and $80\frac{1}{2}$ tons; find the number of units of acceleration in the acceleration of gravity (which is 32·2 sfas); and the number of units of power in 128 H. P.

42. A train of mass 161 tons is being dragged, at a place where the acceleration of gravity is 32·2 sfas, by an engine of 189 H. P. on a level line, the resistance due to friction being 15 lbs. weight per ton of mass. If the velocity at a particular time be 25 miles per hour; find the acceleration at that time.

SECTION IX.

EQUATIONS OF MOTION.

189. In general we shall speak of

t units of Time (second, etc.);		
l	„	Length described uniformly;
x	„	Length not so described;
m	„	Mass (pound, gramme, etc.);
a	„	Area (square foot, etc.);
c	„	Volume (cubic centimetre, etc.);
d	„	Density;
v	„	Velocity (fas, cas, etc.);
$\boldsymbol{a}$	„	Acceleration (sfas, scas, etc.);
k	„	Momentum (fasp, casgram, etc.);
η	„	Kc. Energy (faspen, casgrammen, etc.);
p	„	Force (poundal, dyne, etc.);
i	„	Impulse (bim, cim, etc.);
w	„	Work (footpoundal, erg, etc.).

Also g sfas, scas, etc., will be the acceleration of gravity; and ρ the specific density of any mass.

190. The equations of motion with which we have to deal

are those which are concerned with the rectilinear movement of a single heavy particle.[1]

These equations are derived directly from the laws of motion.

191. Short forms of the three laws are:

Law 1. All mass has the property of Inertia.

Law 2. Force has its equivalent in Momentum.

Law 3. The Work Done by a force has its equivalent in Energy and Heat.

192. Some of the following equations have already been established.

193. If there be c units of Volume
in m units of Mass,
and if the density be d times the unit density;
then $m=dc$;[2] . . . . (1)
[not an equation of *motion*, but one that will be often useful.]

194. If a body moving with
a constant velocity of v units of Velocity,
pass over l units of Length
in t units of Time;
then $l=vt$. . . . . (2)

[1] It will be recognised by those acquainted with the Calculus that the equations are the integrals of $m\ddot{x}=p$.

[2] It should be remembered that, with *British Units*, d is not the *specific density* because *water* is not then of unit density.

195. If a body, with m units of Mass,
is moving with v „ Velocity;
and so has k „ Momentum,
and η „ Kc. Energy;
then $k=mv$; . . . . (3)

$$\eta=\frac{1}{2}mv^2. \quad \ldots \quad (4)$$

constant

196. Now, by the 2nd Law, we measure a force by the momentum produced in a given time.

If there be required p units of Force
to impart k „ Momentum
in t „ Time;

then since

1 unit of force imparts in 1 unit of time 1 unit of momentum;
∴ k units „ impart „ 1 „ „ k units „ ;
∴ $\frac{k}{t}$ „ „ „ „ t units „ k „ „ ;

[since the longer the time allowed for imparting the momentum the less the force required;]

$$\therefore\ p=\frac{k}{t};$$

$$\text{or } k=pt; {}^{*)} \quad \ldots \quad (5)$$

which gives the number of units of momentum imparted in a given time by a given constant force.

*) [illegible]

197. The effect of force is, of course, to impart acceleration to a mass; by 196 force produces momentum

so if there be required p units of Force
to impress a „ Acceleration
on m „ Mass;

then since

	1 unit of force	impresses	1 unit of acceleration	on	1 unit of mass	;
∴	m units „	impress	1 „	„	m units „	;
∴	ma „	„	a units „	„	m „ „	;

$$\therefore \quad ma = p \qquad \qquad (6)$$

198. Once more, the effect of force is to impart to a mass so much velocity in a given time;

so if there be required p units of Force
to impart v „ Velocity
to m „ Mass
in t „ Time;

then since

	1 u. of force	produces	in 1 u. mass	in 1 u. time,	1 u. velocity	;
∴	m „	„	„ m „	1 „	1 „	;
∴	mv „	„	„ m „	1 „	v „	;
∴	$\frac{mv}{t}$ „	„	„ m „	t „	v „	;

$$\therefore p = \frac{mv}{t};$$

$$\text{or } v = \frac{pt}{m}; \qquad \qquad (7)$$

which gives the velocity produced in a mass by a given constant force acting for a given time.

199. Equation (7) might have been obtained by combining equations (3) and (5).

200. Combining equations (6) and (7), we see that if a body moving under

an acceleration of a units of Acceleration,
acquire v „ Velocity,
in t „ Time;

$$v=at; \qquad (8)$$

which has already been established, as well as the same equation in momentum

$$mv=ma\times t. \qquad (9)$$

201. When the constant acceleration is that of gravity, a is usually denoted by g, and we have

$$v=gt; \qquad (10)$$

an important equation.

202. If a body is already moving with a velocity of V units when the acceleration begins to act, its momentum at any time must be that due to its original velocity as well as that due to the acceleration;

∴ by equation (9)

$$mv=mat+mV. \qquad (11)$$

So if a body,

already moving with a velocity of V fas,
come under an acceleration of a sfas;
and so, after t seconds,
is moving with a velocity of v fas;

$$v=V+at. \qquad (12)$$

203. Equations (5) to (12) determine the *acceleration* of a particle acted on by a constant force; and the *momentum* and *velocity* of a particle acted on for a given time by a

constant force, or subject for a given time to a constant acceleration.

We have now to determine the *length* through which a particle moves under these circumstances.

204. It has already been proved that

if a particle, under a units of Acceleration,
move in t units of Time
through x units of Length;

$$x=\frac{1}{2}at^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

whence, by equation (6),

if a particle of m pounds
acted on by p poundals,
move in t seconds
through x feet;

$$x=\frac{1}{2}\frac{p}{m}t^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

the same equation holding good, of course, for any other system of units.

205. If, when the force begins to act, the particle be already moving with a velocity of V fas, this velocity will continue in addition to that imposed by the force, and equation (14) becomes

$$x=Vt+\frac{1}{2}\frac{pt^2}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

And, under similar circumstances, equation (13) becomes

$$x=Vt+\frac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

206. For a body falling from rest we evidently have

$$x=\frac{1}{2}gt^2. \quad \text{(17)}$$

Or if it starts with V units of velocity,

$$x=Vt+\frac{1}{2}gt^2. \quad \text{(18)}$$

If a body be thrown upwards with V units of velocity the acceleration of gravity is against it; and so, then,

$$x=Vt-\frac{1}{2}gt^2. \quad \text{(19)}$$

207. Previous equations for determining the velocity, or the length passed, have also involved the time.

The 3rd Law enables us to determine equations not involving the time.

If a particle of m grammes,
moving under the action of p dynes,
pass over x centimetres,
and acquire v cas;

then, by the 3rd Law,

$$px=\frac{1}{2}mv^2; \quad \text{(20)}$$

the same equation holding good, of course, for any other system of units.

208. We may corroborate equation (20) by obtaining it from equations (7) and (14).

209. The following equations are left for the student to *enunciate* and *prove*,

(i.) from equation (20),

(ii.) by deduction from previous equations:

$$\frac{1}{2}v^2 = ax; \quad \ldots \quad (21)$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mV^2 + px; \quad \ldots \quad (22)$$

$$\frac{1}{2}v^2 = \frac{1}{2}V^2 + ax. \quad \ldots \quad (23)$$

210. Supposing a mass of m pounds to be moved upwards against gravity by a constant force of p poundals;
then, there is a constant force of p poundals upwards and a constant force of mg poundals downwards;
the resultant is, obviously, a constant force of $p-mg$ poundals;
and $\therefore$ we derive from the 3rd Law, the equation

$$\frac{1}{2}mv^2 = (p-mg)x.$$

Or we may look at it in this way:

By Law 3, the work done by the force is equal to the kinetic energy and the work done against gravity

$$\therefore \; p \times x = \frac{1}{2}mv^2 + mg \times x;$$

or, as before,

$$\frac{1}{2}mv^2 = px - mgx. \quad \ldots \quad (24)$$

WORKED EXAMPLES.

A. If a mass of 1 ton is projected with a velocity of a mile a minute, and is acted on in the line of motion by a retarding force of 20,000 poundals; find its momentum after 10 seconds.

This is an example of equation (11) in the form

$$mv = mV - pt;$$
$$= 2240 \times \frac{1760 \times 3}{60} - 20000 \times 10;$$
$$= 197120 - 200000$$
$$= -2880;$$

$\therefore$ required momentum $= 2880$ fasp backwards, *i.e.* contrary to the direction of projection.

B. A particle is projected vertically upwards with a velocity of $4\frac{1}{3}g$ cas, at a place where the acceleration of gravity is g scas; find the times after which it will be at a height of $8g$ centimetres; and its velocity at that height.

From equation (19)

$$8g = \frac{13}{3}gt - \frac{1}{2}gt^2;$$

whence the times are $\frac{8''}{3}$ and $6''$.

By equation (12)

$$v = V - gt;$$

$\therefore$ velocity is $\pm \frac{5}{3}g$ cas;

according as the particle is ascending or descending.

C. Find the acceleration of a particle which would describe 45 miles in the 2nd minute of its motion from rest.

From equation (13), the distance described between two times is clearly

$$\frac{1}{2}a(t_2^2 - t_1^2).$$

$\therefore$ we have

$$45 \times 1760 \times 3 = \frac{1}{2}a(120^2 - 60^2);$$

$$\text{whence } a = 44;$$

and $\therefore$ the acceleration is 44 sfas.

D. At a place where the acceleration of gravity is 32·12 sfas, a mass of 3 lbs. is projected vertically upwards with a momentum of 210 fasp, and is acted on by a constant upward force of 60 poundals. In what time will the mass reach 94 feet on its downward journey?

From equations (15) and (17) we have

$$mx = mVt - \frac{1}{2}mgt^2 + \frac{1}{2}pt^2;$$

$$\therefore 3 \times 94 = 210t - \left(\frac{3}{2}(32{\cdot}12) - 30\right)t^2;$$

$$\therefore 94 = 70t - (6{\cdot}06)t^2;$$

$$\text{whence } t = \frac{470}{303} \text{ or } 10;$$

the former value gives the time of reaching 94 feet on the upward journey; the latter value shows that the mass will reach the same point on its downward journey in 10 seconds from the time of projection.

E. If the application of a certain acceleration reduces the

speed of a mass from 50 fas to 10 fas, in a mile; in how much further would the mass come to rest?

By equation (23)

$$\frac{1}{2}(50)^2-\frac{1}{2}(10)^2=a\times 5280;$$

and by equation (21), if y be the required distance,

$$\frac{1}{2}(50)^2=a(5280+y);$$

from these two equations we obtain $y=220$; and so the mass would come to rest in another one-third of a furlong.

F. A mass of 1 kilogramme impinges with a velocity of 50,000 cas on a slice of resisting material in which, if thick enough, the mass would lose all its velocity in penetrating 25 centimetres. If the slice be but $\frac{1}{4}$ centimetre thick, find the momentum lost by the mass in passing through the slice.

If p dynes be the constant retarding force, and V cas the velocity still to be lost after passing through $\frac{1}{4}$ centimetre; we have, by equations (20) and (22),

$$\frac{1}{2}\cdot 1000\cdot(50000)^2=p\times 25;$$

$$\text{and } \frac{1}{2}\cdot 1000\cdot(50000)^2-\frac{1}{2}\cdot 1000\cdot V^2=p\times\frac{1}{4};$$

$$\text{whence } V=50000\sqrt{\frac{99}{100}};$$

$\therefore$ no. of casgram of momentum lost

$$=1000\times 50000-1000\times V;$$

$$=250000 \text{ approximately.}$$

Examples.—IX.

1. If a mass of 1000 kilogrammes is projected with a velocity of 30 metres a second and is acted on in the line of motion by a retarding force of 300 million dynes; find its momentum after 10″.

2. A shot being fired vertically upwards with a velocity of 1610 fas at a place where the acceleration of gravity (supposed constant throughout the motion) is 32·2 sfas; find the time that would be occupied in passing between two points, one 14,490 feet, the other 25,760 feet, from the ground; neglecting the resistance of the air.

3. A heavy particle, of mass m grammes, is projected vertically upwards with a velocity of u cas at a place where the acceleration of gravity is g scas; find the height to which it will rise, the time taken and the work done in rising to this height, the time it will take to fall to the point of projection, and its velocity when it reaches that point.

4. At a place where the acceleration of gravity is 32·12 sfas, a mass of $8\frac{1}{3}$ pounds is projected vertically upwards with a momentum of 400 fasp, and is acted on by a constant upward force of 1 poundal. In what time will it again reach the ground?

5. Find the height to which, neglecting the resistance of the air, a body will rise which starts upwards with a velocity of 1000 fas; the acceleration of gravity being 32·2 sfas.

6. At a place where the acceleration of gravity is 980 scas, a mass of 1 gramme has been falling for 50 seconds; what force would be required to stop it in $600\frac{1}{4}$ centimetres?

7. The sum of the masses of two bodies, hanging by a massless string which passes over a smooth massless pulley, is 7644 grammes. Determine the greater mass when, at a place where the acceleration of gravity is 980 scas, it acquires from rest in 13 minutes a velocity of 400 cas.

8. For how many seconds has a body been falling from rest, when the number of centimetres that it falls in the last second of its fall is equal to $22\frac{7}{8}$ of the number of fas in its final velocity; [a foot being taken as equal to $30\frac{1}{2}$ centimetres]?

9. Find in sfas the acceleration of a particle which describes 27 feet in the 5th second of its motion from rest.

10. At a place where the acceleration of gravity is 980 scas, a mass of 1 kilogramme is projected vertically upwards with a momentum of two million casgram, and is acted on by a constant upward force of 800,000 dynes. In what time will the mass reach 110 metres on its upward journey?

11. A body is projected upwards with a velocity of 161 fas; how high will it rise; and in what time will it reach its greatest height? [The acceleration of gravity is 32·2 sfas.]

12. At a place where the acceleration of gravity is 32 sfas, a mass of 1 pound penetrates a soft mass offering a uniform resistance with a velocity of 32 fas; and loses all its velocity in penetrating 1 foot. Find the resistance in pounds.

---

13. At a place where the acceleration of gravity is 981 scas, two masses, of 2905 and 2000 grammes, are suspended at the ends of a massless string which, passing over a smooth massless pulley, hangs vertically. If a downward velocity of 190 cas be given to the larger mass, what will be its velocity after 10″?

14. A particle falls from rest from a point A at a place where the acceleration of gravity is 980 scas, and a second afterwards a second particle is let fall from a point 980 centimetres below A. When and at what distance below A will the two particles be together?

15. A constant force acts on a particle for a minute and then ceases. In the next minute the particle describes 15 miles. Compare the acceleration of the force with that of gravity at a place where with British units $g = 32{\cdot}12$.

16. A mass of 7 kilogrammes is projected on ice with a velocity of 2400 cas, and in half a minute travels 360 metres. Find the force of the friction.

17. A ball penetrates a block with a velocity of 50,000 cas, and loses all its velocity in penetrating 25 centimetres; in what time is it reduced to rest, supposing the retardation constant?

18. Find the number of cwt. in a constant force which (at a place where the acceleration of gravity is 32 sfas) would in half a mile stop a train of 100 tons moving at the rate of 27 miles an hour.

---

19. A mass at rest is acted on by a force of a cwt. to the ton of mass; in what time will it attain a velocity of 21·9 miles an hour, at a place where the acceleration of gravity is 32·12 sfas?

20. A ball is projected vertically upwards with a velocity of 322 fas at a place where with British units g is 32·2; and when it has passed half the time of its ascent, a second ball is projected from the same point with the same velocity. When will they meet, and at what distance from the ground?

21. At a place where the acceleration of gravity is 32 sfas, a

body is dropped from a height of 144 feet, and 1″ afterwards another body is projected vertically downwards from the same point. What must be the velocity of projection that the two bodies may reach the ground together?

22. At a place where the acceleration of gravity is 32·2 sfas, through what length will a mass of $\frac{1}{4}$ ton move in 10″ under the action of a force of 10 pounds?

23. In question 17, how many cas of velocity does the ball lose in penetrating 1 centimetre?

24. A train of 90 tons is travelling at the rate of 30 miles an hour, and, the breaks being put on, is stopped in a furlong. Find the number of sfasp exerted by the break.

---

25. Find the acceleration of the centre of mass of two particles, of mass m and m' pounds, moving vertically and connected by a massless string passing over a smooth massless pulley.

26. At a place where the acceleration of gravity is 32 sfas, a balloon ascends with a uniform acceleration of 11 sfas, and at a height of 352 feet a stone is dropped. How long will the stone continue to rise; and after what further time will it reach the ground?

27. From the same point, and at the same time, two particles A and B are projected vertically downwards, and a third C is let fall. A reaches the ground in half the time that B takes, and C is but half way down when B reaches the ground. Find the relation between the initial velocities of A and B.

28. A mass travelling with a velocity of a kilometre a minute (at a place where the acceleration of gravity is 980 scas) is suddenly subject to a resistance of 5 kilogrammes weight per mass of 1000 kilogrammes; in what time will it travel $1421 \cdot \dot{6}$ metres?

29. If a bullet in passing through a board has its velocity reduced from 50,000 cas to 40,000 cas; through how many boards would it pass before it came to rest?

30. A bullet of mass 16 grammes leaves a gun whose mass is $4\frac{1}{2}$ kilogrammes with a velocity of 325 mas (metres a second). If the 4 grammes of powder leave the gun with an average velocity of 275 mas; find in terms of the weight of a kilogramme the constant force necessary to just allow the gun to kick through 8 centimetres, at a place where the acceleration of gravity is 980 scas.

---

31. A mass of 1 gramme, projected in the direction opposite to the action of a constant force, has after $\frac{1}{3}''$ and $1''$ momenta respectively of 673 and 19 casgram. Find the force and the velocity of projection.

32. A shot being fired vertically upwards with a velocity of 320 fas at a place where, with British units, $g=32$; find the time occupied in passing between two points, one 576 feet, the other 1024 feet, from the ground.

+ 33. A person drops a stone into a well, and after $3''\cdot675$ hears it strike the water; find the depth to the surface of the water. [Neglect the resistance of the air; and take the acceleration of gravity as 32 sfas, and the velocity of sound as 1120 fas.]

34. At a place where the acceleration of gravity is 32·2 sfas,

a mass travelling with a velocity of 40 miles an hour is suddenly subject to a resistance of 10 pounds per ton; in what time will it travel 6005 feet?

35. Find the momentum acquired by a mass of 12 stone falling a height of 3 yards at a place where the acceleration of gravity is 32 sfas.

36. A mass of a quarter of a ton is acted on by a force equal to the weight of 1000 tons while it moves from rest through 10 feet, at a place where the acceleration of gravity is 32 sfas. Find the velocity acquired; and if the force now cease and the necessary constant resistance of 10,000 tons be applied so as to stop the mass in 1 foot; find the time in which it will be so stopped.

SECTION X.

THE SECOND LAW OF MOTION (*Continued*).

Composition of Motion and Force.

211. HITHERTO we have considered a mass moving with so much velocity, or momentum, or kinetic energy, or under the action of force, in *one direction* only.

212. A train moving with a definite velocity has been stopped by friction in a given time.

A force has acted on a recoiling gun for a given space, and there has been so much loss of kinetic energy.

A blow has been given to a "Try your Strength"; or a mass has fallen under the action of the earth's attraction.

Or, more complicately, a blow has projected a mass horizontally; and then, the constant force of the earth's attraction has compelled the mass to describe a parabola.

In these cases there has been no complication of directions.

213. But, clearly, a mass may be given momentum or kinetic energy, and may be under the influence of force, in many different directions.

Thus a ball moving in one line may be struck by another

moving in a line inclined to the first; and it will be a problem to discover its subsequent momentum and kinetic energy.

The moon's path is mainly determined by the velocity it already has, and by two forces (the attractions of the earth and sun) inclined to one another at a varying angle.

If a pendulum be formed of a heavy particle and a massless string, the particle (if in a vacuum) swings under the action of two forces, its weight and the pull on the string.

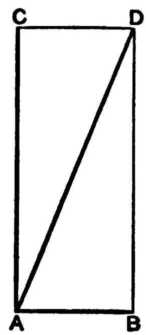
214. When two velocities in different directions have been given to a particle, it will move, of course, with a definite velocity in some one direction; which (by law 1) it will retain if no further force act upon it. One of the problems of this section is to determine this *Resultant Velocity.*

215. If two forces in different directions act on a particle, each force impresses (in the time considered) so much velocity in its own direction; and the result is a velocity in a third direction. This, clearly, may be looked upon as if due to a third force in this third direction. Another of the problems of this section is to determine this *Resultant Force.*

216. *Per contra,* in problems it is often useful, when one velocity or one force is given, to consider, the velocity or force as if it were the resultant of two others in different directions. The *Resolution of Velocity* and the *Resolution of Force* are also problems of this section.

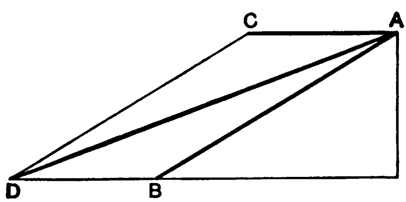
217. PARALLELOGRAM OF VELOCITIES.—**If a mass is constrained to move with two constant velocities in two constant directions: and if these velocities be represented by two adjacent sides of a parallelogram; the concurrent diagonal of the parallelogram will represent in magnitude and direction the actual velocity with which the mass is moving.**

218. If while a ship is moving North, along *AC*, through 36 feet, a man on the deck moves East, along *AB*, through 15 feet; his position at the end of the motion is at *D*, 39* feet from *A*, at the other extremity of the diagonal *AD* of the parallelogram described on *AC* and *AB*.



And so, generally, it is obvious that if a point move in two directions through two lengths, its ultimate place will be at the extremity of the parallelogram formed on the two lengths.

Thus if a mass slide down an inclined plane from *A* to *B*,



and if the plane move so that *A* comes to *C*; the ultimate position of the mass is at *D*, at the extremity of the diagonal of the parallelogram formed with *AB* and *AC* as adjacent sides.

* $39^2 = 36^2 + 15^2$.

219. It clearly makes no difference to these results if the motion considered takes place uniformly and during the same time.

Suppose that 7″ is the time during which the ship moves uniformly over its 36 feet; and that the man moves through his 15 feet uniformly in the same time.

Then clearly at the end of 1″, the two ~~movements~~ velocities are $\frac{36}{7}$ and $\frac{15}{7}$ feet; and we get a similar and similarly placed parallelogram; and the man has actually moved through $\frac{39}{7}$ feet in the direction *AD*. So at the end of any other time.

So the velocity of the ship being $\frac{36}{7}$ fas, and that of the man on the deck $\frac{15}{7}$ fas; his actual velocity is $\frac{39}{7}$ fas.

220. And so with the inclined plane. If the mass could slide down uniformly in t'', while the plane moved (also uniformly) through a distance equal to *AC* in t''; *AB* and *AC* would represent the velocities of the mass owing to the two causes; and its actual velocity would be represented by *AD* (in magnitude and direction).

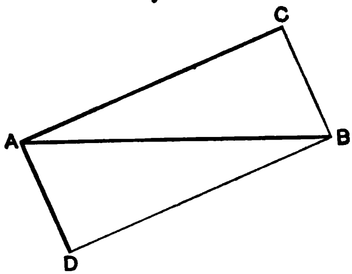
221. So in the general case.

222. Resolution of Velocity.—It is clear that just as when a mass has two different velocities given to it the result is one definite velocity; so **if a mass has a definite velocity in a definite direction, it may be assumed** (if con-

venient for the purposes of the problem) **to have two separate velocities in any convenient directions.**

Thus supposing a mass to have a velocity of 13 cas in the direction AB; then if AC, AD be two directions at right angles to one another, such that

$$\tan BAC = \frac{5}{12};$$



the mass may be supposed to have a velocity of 12 cas in the direction AC, and of 5 cas in the direction AD.

223. When the velocities are not constant, it will be necessary and sufficient to consider the mass as moving in jerks, the velocities being constant during each successive moment.

The Parallelogram of Velocities will then give the actual velocity during the moment.

224. Relative Velocity.—In the above problems the mass considered has had two velocities in different directions, and by a geometrical construction we have found its actual velocity.

The man on the deck had a velocity of $\frac{15}{7}$ fas owing to his own exertion, and a velocity of $\frac{36}{7}$ fas owing to the motion of the ship; and his true velocity was a compound of these.

But in many problems what is given is the velocity of two masses in different directions; and it is required to find the velocity of one *relative* to the other.

225. Definition.—If two masses, A and B, are in motion; and an additional velocity be impressed on each (the same on each), such that A is brought to rest; then the velocity of B is its *Velocity Relative* to A.

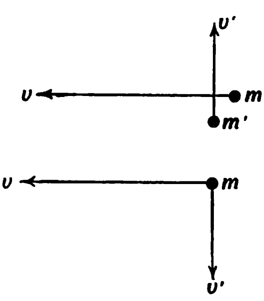
This additional velocity being impressed on each cannot alter the *relative* velocity; and yet gives us a means of determining it.

226. As a fact all motion is relative; and when we speak of a man walking with a velocity of 4 miles an hour, we mean relatively to the earth. His actual velocity in space is enormous.

227. When the masses are moving in the *same straight line*, there is no difficulty with the Relative Velocity; which, clearly, is the *sum or difference* of the velocities.

228. When the directions are not the same, the following is an example of the method employed.

Let m and m' lbs. be moving towards one another, with v and v' fas, in directions at right angles to one another; to find their relative velocity just before impact.



Impress on each of them a velocity of v' fas, in a direction opposite to that in which m' is moving.

We shall now have m' at rest, and m moving with two velocities v and v' fas, at right angles to one another.

These being combined by the parallelogram of velocities,

the resultant gives the velocity of m relative to m', in magnitude and direction.

229. Parallelogram of Forces.—**If two forces (taken as producing velocities constant during a moment) act on a particle, and be represented in magnitude and direction by the adjacent sides of a parallelogram; the concurrent diagonal of the parallelogram will represent in magnitude and direction a third force which would, in the moment, produce in the particle the same momentum as is produced by the two forces.**

230. Take a parallelogram whose two sides are proportional to the forces.

By the 2nd law the forces (being supposed to act in jerks) are proportional to the momenta produced; and (as the mass is the same for each) are therefore proportional to the velocities produced.

So these velocities are proportional to the sides of the parallelogram.

But, by the parallelogram of velocities, these two velocities may be represented by a single velocity along and proportional to the diagonal of the parallelogram.

This could be produced by a single force also along and proportional to the diagonal of the parallelogram.

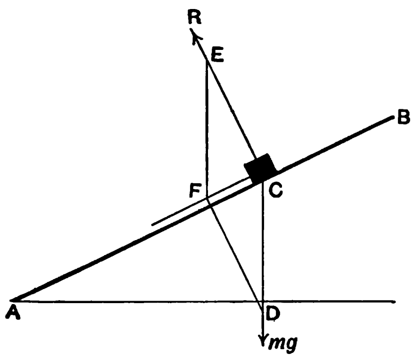
Which force is, accordingly, the resultant of the other two.

231. It is necessary to deal with jerky motion, because the directions of forces will, in dynamics, seldom remain constant.

When they do remain constant the parallelogram of forces may be stated as follows.

232. If *two constant forces,* acting for the same time *in constant directions* upon a particle, be represented in magnitude and direction by two adjacent *sides of a parallelogram*; the concurrent *diagonal* of the parallelogram will represent in magnitude and direction a third constant force (the *Resultant* of the other two) which would *in the same time* produce in the particle the same momentum as is produced by the two constant forces.

233. Thus suppose a smooth inclined plane, and a mass, m lbs., to be sliding down the plane.



Two forces are acting on m; the weight mg poundals, and the pressure R poundals.

explain!

If we mark off CD and CE, so that

$$CD : CE :: mg : R,$$

and complete the parallelogram; CF, the diagonal, will

represent the line of action and magnitude of the resultant force.

In this case, the two forces mg poundals and R poundals are constant, and remain in fixed directions during the time considered.

234. In the case of two constant forces acting for a definite time in constant directions the parallelograms are (as is easily seen) all similar and similarly situated; and the resultant is a third constant force acting for the same definite time along the common diagonal of the parallelograms.

This (as above stated) will not always, nor often, be the case.

235. When there is a variation of the magnitude or direction of either force, the parallelogram of forces can still be held true by considering the forces to act in jerks, and the velocities to be constant during each moment.

In general, the working out of such a problem will require the processes of higher mathematics.

236. The Parallelogram of Forces will clearly maintain in the case of two *Blows* administered at the same instant.

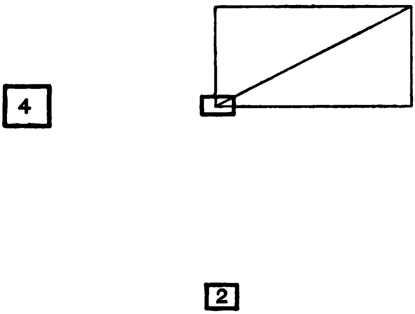
The following is a practical illustration of this.

237. I suspended three kitchen "weights," 1 lb., 2 lbs., and 4 lbs., by thin strings, 10 feet long, from the same point. The weights hung within an inch of the ground.

The 1lb. was placed hanging vertically, and exactly under

its centre was the corner of a sheet of paper, in shape a rectangle whose sides were one double of the other.

The 4 lbs. and 2 lbs. were then drawn back, in the directions respectively of the long and short sides of the rectangle, to a distance of 2 feet from the corner, and let go simultaneously.



They struck the 1 lb. at the same instant, and the 1 lb. swung off in the direction of the diagonal of the rectangle.

As the 4 lbs. and 2 lbs. were equally raised from their lowest levels, their velocities (by the kinetic energy equation) were equal at the instant of striking. And so (their masses being as 2 : 1) one blow was double of the other; and the 1 lb. mass rightly went off in the direction of the diagonal.

238. It is to be carefully noted that the reason why there is a parallelogram of forces, is because of the two truths contained in the second law of motion.

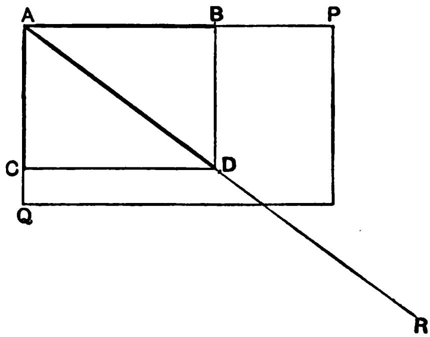
239. *One truth* is in a very important part of this second law which has hitherto been but slightly noticed, viz., that

which declares that the *momentum is in the direction of the force.*

We have really assumed the truth of this in every problem connected with force; but our experience accepts this fact, in the case of *one* force, as an obvious truism.

The importance of this part of the 2nd law is seen when we have more than one force acting. Then the law implies that every force has its full effect *in its own direction,* without any reference to other forces, or to the velocity with which the mass is moving.

240. *The other truth* is that it is the *Momentum* produced in a given time (*and not the Kinetic Energy*) which is proportional to the force.



Let there be a mass of m lbs. at A; and let ABP, ACQ be at right angles.

Let P be a force which acting in the direction ABP produces in m, in t seconds, a velocity of 4 fas; and let Q be a force which acting in the direction ACQ produces in m, in t seconds, a velocity of 3 fas.

If we let the velocities[1] be represented by AB ($=4$ quarter-inches) and AC ($=3$ quarter-inches); the resultant velocity must be represented by AD ($=5$ quarter-inches).

[So far we get without any laws of motion at all; or, at all events, with only the 1st law, which teaches us that force produces velocity, without giving the exact relation.]

If now the second law is not true; and if it is the *kinetic energy* developed in a given *time* which is proportional to the force; then P and Q are proportional respectively to 16 and 9 faspen.

Let P and Q be represented by AP ($=16$ quarter-centimetres) and AQ ($=9$ quarter-centimetres); the resultant force R must be represented by AR ($=25$ quarter-centimetres) in the direction ADR; for that clearly represents the actual kinetic energy at the moment.

But it is clear that AR is not the diagonal, either in magnitude or direction, of the parallelogram formed on AP and AQ.

241. We may say that the Parallelogram of Forces is really self-evident when, by means of the laws of motion, we have established the two following truths:—

1*stly*, That the actions on a particle of two forces are entirely independent.

2*ndly*, That two forces acting during the same moment are proportional to the velocities produced in the particle.

The 1st comes from that part of the 2nd law which declares that the action of a force is only in its own

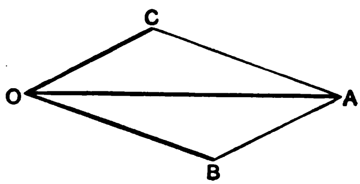
[1] Not the forces here; only the velocities.

direction; the 2nd from that part which states that the amount of momentum produced in a given time is proportional to the force.

242. Resolution of Force.—Just as the effect of two forces may be considered as the effect of one; so **the effect of one force may be considered as if it were the effect of two others in different directions**; and this principle is very largely used in the solution of Dynamical problems.

There are also many instances of one force producing two or many. A wedge is a simple case; the push at the back of the wedge produces forces at each side of the wedge.

243. When we have taken a line *OA* to represent the given force, we may form any parallelogram *OCAB* of which *OA* is the diagonal; and the effect, during a moment, on the mass at *O* of the force in the direction *OA*, will be the same as the effect, during the moment, of two forces proportional to and in the direction of *OB* and *OC*.



For example, students of analytical conics will see that thus great simplifications of complicated problems, concerning forces in one plane acting on a particle, may be secured by resolving all the forces by means of rectangles whose sides lie along two rectangular axes having the particle at the origin. All the forces could then be added together into two forces acting along the axes; and could then, of course be compounded into one single force.

244. To determine the value of the resultant of two forces of P and Q poundals acting for a moment at an angle of $\theta°$.

Let OB, OC (see the figure on p. 147) represent the two forces in direction and magnitude;
therefore OA represents the resultant (R poundals) in direction and magnitude;

$$\therefore\ OB : OC : OA :: P : Q : R;$$

Now, by Trigonometry,

$$\begin{aligned} OA^2 &= OB^2 + AB^2 - 2OB \cdot AB \cos OBA\,; \\ &= OB^2 + OC^2 - 2OB \cdot OC \cos(\pi - \theta)\,; \\ &= OB^2 + OC^2 + 2OB \cdot OC \cos\theta\,; \\ \therefore\ R^2 &= P^2 + Q^2 + 2PQ\cos\theta. \end{aligned}$$

This equation will enable us to determine any one of the four quantities P, Q, R, θ, when the other three are given.

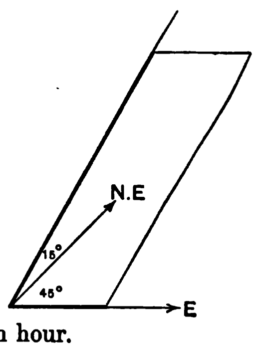
WORKED EXAMPLES.

A. In a current which flows to the E. at the rate of $2\frac{1}{2}$ miles an hour, a steamer is going with its head 15° to the north of N.E. at the rate of 8 miles an hour; find the true velocity of the steamer.

The angle between the two directions is 60°;

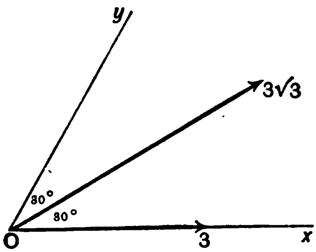
If v miles per hour be the true velocity;

$$\begin{aligned} v^2 &= (2\tfrac{1}{2})^2 + (8)^2 + 2(2\tfrac{1}{2}) \times 8 \times \cos 60\,; \\ &= \frac{25}{4} + 64 + 2 \cdot \frac{5}{2} \cdot 8 \cdot \frac{1}{2}\,; \\ &= \frac{361}{4}\,; \end{aligned}$$



$\therefore$ velocity is $9\frac{1}{2}$ miles an hour.

B. If a point is subject to two velocities, one of 3 fas in a certain direction Ox, the other of $3\sqrt{3}$ fas, in a direction inclined at an angle of 30° to Ox; what is its velocity in a direction inclined at 60° to Ox?

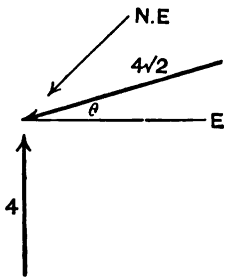


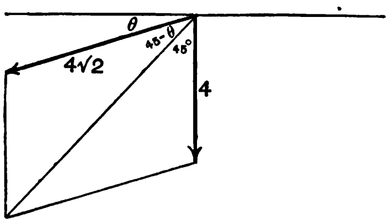
Velocity in fas along Oy (the rest being perp. to Oy)

$$=3 \cos 60^\circ + 3\sqrt{3} \cos 30^\circ;$$

$$=6.$$

C. In a wind moving at the rate of $4\sqrt{2}$ miles an hour, a man is walking North at the rate of 4 miles an hour, and the





wind seems to come from the N.E. What is its true direction?

Let the true direction be Θ° to the north of E.

Take a particle of dust; and, to find the velocity relative to the man, impress on the particle the velocity of the wind and one equal and opposite to that of the man.

It is given that the resultant of these two velocities is in a direction N.E. to S.W.

We clearly have

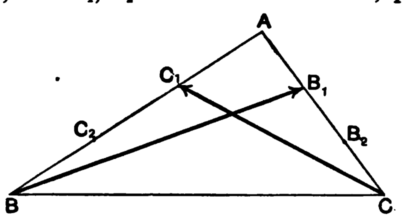
$$4 : 4\sqrt{2} :: \sin(45^\circ - \theta^\circ) : \sin 45^\circ;$$
$$\text{whence } \theta^\circ = 15^\circ.$$

D. The point of application of a force, which preserves its direction, moves a mile in a minute in a line inclined to the force at an angle of 60°. If the rate of working is a Horse-Power, what is the force?

If the force be P pounds weight,

$$\text{it does } P\cos 60^\circ \times 5280 \text{ foot-pounds in } 1';$$
$$\therefore\ P = \frac{33000 \times 2}{5280} = 12\tfrac{1}{2}.$$

E. If B_1, B_2 be the points of trisection of the side AC of a triangle ABC; and C_1, C_2 those of the side AB; prove that the



resultant of forces represented at any instant in magnitude and *position* by BB_1 and CC_1 will be equal and opposite, at the instant, in magnitude and *direction* to the resultant of forces represented in magnitude and *position* by AB_2 and AC_2.

The principle of transferability of force implied!

It is easily seen that the force represented by BB_1 is equivalent to two represented in magnitude and direction (not position) by CB_1 and BC.

Artificial!

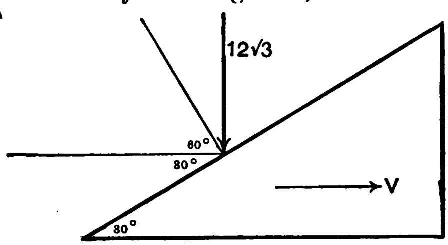
Similarly CC_1 may be replaced by BC_1 and CB.

$$\therefore\ BB_1 \text{ and } CC_1 \equiv CB_1,\ BC_1,\ BC, \text{ and } CB.$$

The two last of these four cancel one another, and the other two are opposite to AB_2 and AC_2.

F. A piece of putty, massing 1 lb., falls with a velocity of $12\sqrt{3}$ fas on a smooth inclined plane, massing 2 lbs., whose angle of inclination is 30°, and which can move freely on a smooth horizontal plane. Find the force of the blow at right angles to the inclined plane, and the initial velocity of the plane.

The two masses will initially move together;



Let V fas be the velocity of the plane, and therefore also the horizontal velocity of the putty.

The momentum of the blow perpendicular to the inclined plane being B fasp;

$$B = 1 \times 12\sqrt{3} \times \cos 30° - 1 \times V \cos 60°;$$

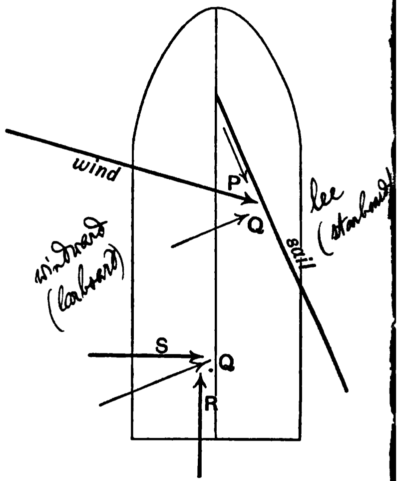
$$\text{and } B \cos 60° = 2V;$$

whence, the blow is 16 fasp;

and the velocity 4 fas.

X G. Show how a ship can tack against the wind.

Place the sail on the opposite side to that on which the wind is, and inclined to the ship at a smaller angle than is the wind.



The force of the wind may be resolved into P and Q poundals, respectively along and at right angles to the *sail.*

P has no effect.

Consider the effect of Q on the ship, by resolving it into R and S poundals along and at right angles to the *ship.*

S has little or no effect because of the resistance of the water.

The only effective force therefore is R; which, of course, tends to send the ship forward.

H. Show that the proof of the parallelogram of forces ordinarily given in Statics assumes the 2nd law.

The proof, in assuming that a force can be represented by a straight line, assumes that a force is proportional to the velocity produced in a given time (and not *e.g.* to the kinetic energy); and that the effect of a force is in its own direction.

EXAMPLES.—X.

1. A marble is rolled on the floor of a lift with a velocity of $\frac{18}{11}$ miles per hour, while the lift rises with a velocity of 1 fas. Determine the actual velocity of the marble.

2. A point is moving at the rate of $52\frac{8}{11}$ miles per hour; and its velocity is resolved into two velocities at right angles to one another. If one of these velocities is 56 fas, what is the other?

3. Two men are walking towards each other with velocities of 3 and $3\sqrt{2}$ fas in directions inclined at an angle of 45°. What is their relative velocity and direction at the moment of collision?

4. Two forces of 5 lbs. and 16 lbs. are inclined to one another at an angle of 60°, and preserve their directions. The point of meeting of these directions moves along the line of the resultant force through 33,000 feet in 1 minute. Compare the rate of working with a Horse-Power.

5. *ABCD* is a parallelogram and *E* a point on the side *AB*. What line represents the resultant of forces which, for the moment, are represented in position and magnitude by the lines *AD*, *DE*, *EC*, and *CB*?

6. A mass of $\sqrt{3}$ lbs., moving with a velocity of 3 fas, is struck (as it moves) a blow at right angles to its line of motion. If it is deflected through an angle of 30°, what is the momentum of the blow?

7. Why is it difficult (at a railway station) when two trains are in, for a passenger in one to tell whether it is his or the other that has begun to move?

8. Show that there is no parallelogram of kinetic energies.

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9. From the window of a train moving at the rate of 30 miles an hour, a bottle is thrown with a velocity of 33 fas at right angles to the direction of the train's motion. With what velocity will it begin to move?

10. A steamer is going due E. at the rate of 12 miles an hour; and a man on the deck walks (in a direction inclined to the vessel's length at an angle of $\cos^{-1}\frac{12}{13}$) with an apparent velocity of 13 miles an hour. What pace is he going relatively to the deck, and what is his true direction?

11. Two trains start from the same point and travel along lines, inclined at an angle of 120°, with velocities of 30 and 50 miles an hour; what is their relative velocity?

12. Two forces of 17 and 208 poundals act on a mass of 31 lbs. The two forces are inclined at an angle of 60°; find the acceleration produced.

13. *ABCDE* is a semicircle whose diameter is *AE*; and *AB*, *BC*, *CD*, *DE* are chords of equal length. Show that the resultant of four equal forces represented, for a moment, in magnitude, position, and direction by *AB*, *BC*, *CD*, *DE*, passes through the intersection of the tangents at *B* and *D*; and find its magnitude and direction.

14. A mass of 1 kilogramme moves round a regular hexagon with a constant velocity of 9 kilometres an hour; what must be the impulse which acts at each vertex?

15. Show from the second law the "physical independence of forces."

16. If a blow were proportional to the kinetic energy exhibited, show that the parallelogram of blows would not maintain.

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17. The resultant of two velocities one of 7 fas, and the other of $22\frac{1}{2}$ miles an hour, is a velocity of 37 fas; find the angle between the velocities.

18. If a man of 10 stone jump out of a carriage going at the rate of $7\frac{1}{2}$ miles an hour, his apparent direction making an angle of 60° with that of the carriage; and if he strike a lamp-post with a momentum of 3080 fasp; in what direction does he jump?

19. A man walks at $4\frac{1}{2}$ miles an hour at an angle of 120° with a railway line, and estimates that a train which started from the same point, and at the same time as he did, is travelling relatively to him at the rate of $30\frac{1}{2}$ miles an hour; what is its actual speed?

20. Two horizontal forces equal to the weights of 3 lbs. and 5 lbs. act, at an angle of 60°, on a mass of 12 lbs. lying on a rough plane. The force of friction being one-twelfth of the weight, compare the acceleration with that of gravity.

21. Forces act for a moment along the perpendiculars drawn from the vertices of a triangle to the opposite sides. Find their relative magnitudes that for the moment there may be no acceleration.

22. Three equal perfectly elastic balls are, A at rest, and B and C moving down upon A with velocities of 30 and $30\sqrt{3}$ fas in directions making a right angle with one another. B and C strike A at the same moment, and are brought to rest by their collision; find at what inclination to B's direction, and with what velocity, A moves.

23. Enunciate the parallelogram of momenta, and the parallelogram of accelerations.

24. If forces were proportional to the kinetic energy produced by them in a given *time*, show that the sum of two forces in a straight line would not be equal to the total force.

25. A man jumps from a train moving at the rate of $22\frac{1}{2}$ miles an hour in a direction inclined to the train at an angle of 60°. His velocity if the train had been still would have been 7 fas; what is his actual velocity?

26. Two velocities of 85 fas and $7\frac{1}{2}$ miles an hour compound into a velocity of 91 fas; find the angle between them.

27. Rain is falling vertically with a velocity of 66 fas; but to a person in a train it appears to be inclined at an angle of 45° to the vertical. At how many miles per hour is the train moving?

28. The point of application of a force of 37 lbs. (which preserves its direction) moves 50 feet in 1 second along the line of one of its components. The other component is 7 lbs., and the two components are inclined together at an angle of 60°. Compare the rate of working with a Horse-Power.

29. *ABC*, *DBC* are two triangles (of altitudes h_1 and h_2 feet) on the same base *BC*, and on the same side of it; and forces act for a moment along and proportional to *BA*, *AC*, *CD*, *DB*, *BC*. Find the magnitude and position of the resultant.

30. A cricket ball, massing 6 oz., and moving horizontally with a velocity of 30 fas, is struck a horizontal blow of 27 fasp in a direction at right angles to its line of motion; with what velocity will it begin to move? And how far will it go, if it be struck at a height of 2 ft. 3 ins. from the ground, and if after reaching the ground it rolls in grass which offers a resistance equal to the weight of $6\frac{1}{2}$ oz? [Take $g=32$, and neglect the resistance of the air.]

31. Explain why in driving in a Hansom cab with heavy rain but no wind, the rain beats into the cab.

32. If two forces be inclined at an oblique angle, show that if they were proportional to the kinetic energy produced in a given time, the parallelogram of forces would not maintain in magnitude or direction.

---

33. Two persons A and B, 275 yards apart, start walking, A towards the position which B leaves at the rate of 5 miles an hour, B in a line perpendicular to AB at $3\frac{3}{4}$ miles an hour; in what time will they be nearest together?

34. When a mass describes a circle, of radius a feet, round a centre of force in the centre of the circle, with a constant velocity V fas; find the constant acceleration to the centre.

35. A stone is thrown horizontally with a velocity of 22 fas at a carriage moving in a line inclined at an angle of 60° to the line of the stone. The stone enters through the centre of the side window, and passes out exactly opposite through the centre of the window on the other side. With what velocity is the carriage moving?

36. At a place where the acceleration of gravity is 32·2 sfas, two forces, one of 15 poundals and the other of 5 lbs., act, at an angle of 60°, on a mass of 13 lbs. How long will it take for the mass to acquire a velocity of $32\frac{1}{2}$ fas?

37. If two forces are inclined at an angle of θ; find the least value of θ that the resultant of the two forces may be equal to the less force.

38. A mass of 1 lb. moves round a regular figure of n sides with a constant velocity of 1 fas; what must be the impulse which acts at each vertex?

39. Why did not the ancients discover that the earth moves round the sun?

40. Show that if the number of units of force were given by the number of units in any power of the velocity (produced in a given time) except the 1st, the parallelogram of forces (inclined at a right angle) would not hold in magnitude or direction.

SECTION XI.

THE THIRD LAW OF MOTION (*Continued*).

Gravity and Friction.

(Pendulum, Atwood's Machine, Inclined Plane.)

245. THE Third Law tells us that there are resistances to the full exhibition of energy by a force.

If there were no resistances the whole work done by a force would be exhibited as energy.

But there are resistances of three kinds: those arising from Gravity, those from Molecular Forces, and those from Friction.

The consideration of the resistances arising from Molecular Forces does not come into the scope of this work; but it is proposed to say a few words on the determination of the forces of Gravity and Friction.

246. GRAVITY.—One thing is obvious about the force of gravity, and that is that masses (notwithstanding the resistance of the air) fall quicker and quicker as they descend.

[A fall from the roof of a house will kill a man, while a fall of a few feet will but bruise him.]

This shows that there is a force always acting, as the addition of velocity can (by the 1st law) only be caused by force.

247. But it is obvious that masses fall far too quickly for any direct experiments to be made upon them.

If a dweller in Edinburgh (where the acceleration of gravity is 32·2 sfas) should set up a line 16·1* feet high in a vacuum, and should drop a mass along the line; he would not be able to be sure that exactly a second passed during the drop.

Various methods therefore have been devised by which g may be obtained indirectly. The machines to be here described are the Pendulum, Atwood's Machine, and the Inclined Plane.

248. The Pendulum.—The best method of determining the force of gravity is by means of the Pendulum.

249. We are all familiar with the Pendulum of a clock; a thin rod with a bob at or near the end.

This, though apparently a very simple thing, is, in reality, complicated.

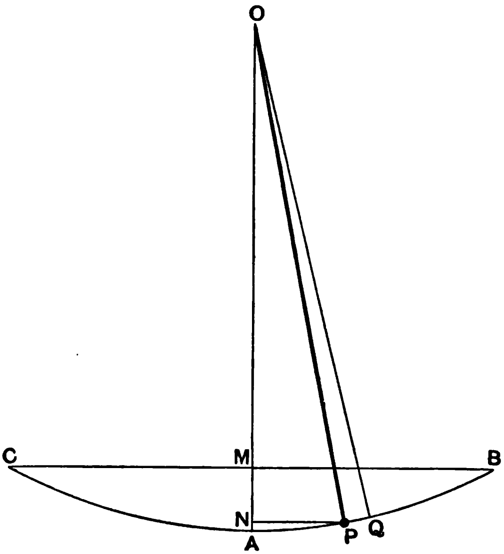
250. Definition.—What is called a *Simple Pendulum*, and that with which we have to do, is a *heavy particle* at the end of a *massless rod.*

Such a pendulum does not exist; but a heavy ball at the end of a thin rod or string is very approximately a simple pendulum.

* The velocity at the beginning of the second being zero, and at the end 32·2 fas; the average velocity is 16·1 fas, and the space described in the second 16·1 feet.

251. The following is the proposition on which the determination of g depends.

Let OP* be a pendulum of length l feet; at a place where the acceleration of gravity is g sfas;



and let t'' be the time in which it beats or oscillates (*i.e.* the time in which it swings from B to C, its highest points on either side).

To show that if the oscillations be small, so that the chord AB may be taken for the arc AB

$$t=\pi\sqrt{\frac{l}{g}}.$$

* The figure is drawn necessarily very much out of scale. Compared with the arc AB, OP is very much too short. If OP, for instance, were 3 or 4 feet long, the arc CAB would be much more nearly a straight line.

252. Let OA be vertical, so that A is the lowest point to which the pendulum-bob falls.

By the 3rd law, the work done by gravity in bringing the bob from B to P is equal to the kinetic energy gained (work done against the friction of the air being neglected);

$$\text{or } mg \times MN = \frac{1}{2} mv^2 ;$$

(if v fas be the velocity of the bob at P.)

Now if B be joined to the other extremity of the diameter through A, then, by Euclid vi. 8, cor.,

$$\text{chord } AB = \sqrt{2l.AM} ;$$

similarly

$$\text{chord } AP = \sqrt{2l.AN} ;$$

$$\therefore v^2 = 2g \times MN ;$$

$$= \frac{g}{l}(2l.AM - 2l.AN) ;$$

$$= \frac{g}{l}(AB^2 - AP^2).$$

Let $AP = AB \sin \Theta$; where Θ is an angle which varies from zero to a right angle;

$$\therefore v^2 = \frac{g}{l} AB^2 \cos^2\Theta.$$

Let PQ be a very small arc,

and let

$$AQ = AB \sin(\Theta + i) ;$$

where i is, of course, a very small part of the right angle to

which Θ extends; and is the portion of the right angle represented by PQ.

$\therefore PQ = \text{arc } AQ - \text{arc } AP;$

$= \text{chd. } AQ - \text{chd. } AP$ (by hypothesis);

$= AB\,(\sin(\Theta + i) - \sin\Theta);$

$= AB\,(\sin\Theta\cos i + \cos\Theta\sin i - \sin\Theta);$

$= AB\cos\Theta \times i;$

[i being so small that we may take cosi equal to unity, and the sine of i equal to its circular measure.]

So then the time of describing PQ

$$= \frac{\text{no. of feet in } PQ}{\text{no. of fas in vel.}};$$

$$= \frac{AB\cos\Theta \times i}{\sqrt{\frac{g}{l} \times AB\cos\Theta}};$$

$$= i\sqrt{\frac{l}{g}}.$$

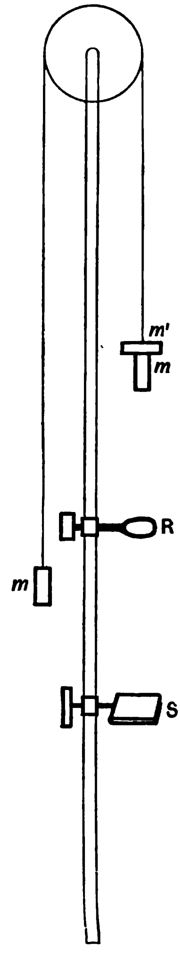
And as therefore the time of describing each small arc is proportional to that part of the right angle which it represents; therefore the whole time is proportional to the right angle;

$$\therefore \text{time from } B \text{ to } A = \frac{\pi}{2}\sqrt{\frac{l}{g}};$$

$$\therefore \text{whole time from } B \text{ to } C = \pi\sqrt{\frac{l}{g}}.$$

253. This remarkable result shows that with small oscillations the whole time of swing will be the same no matter how great the swing.

254. So if a pendulum be set swinging and a note be made of the number of oscillations in a given time, we can find very accurately the time of one oscillation; then, the length of the pendulum being measured, the formula will give us the value of g.



255. It will be noticed that owing to the above result for the time of an oscillation (viz., that the oscillations are isochronous), no allowance has to be made in the experiments for any unevenness in the oscillations.

256. ATWOOD'S MACHINE.—Another method by which it has been sought to determine the force of gravity is by Atwood's machine.

Suppose two equal masses, m and m, to be suspended by a massless string over a massless, frictionless pulley.

They will, of course, remain at rest.

Now if a fresh mass, m', be placed on one of the m's the system will begin to move.

Atwood's contrivance consists in this, that after a given time m passes through a ring R, leaving m' on the ring; and the system keeps moving (by the 1st law) with equable velocity, until m strikes a stage S.

257. Suppose, now, that the system is so arranged as to move for 1″ before reaching R, and for 1″ after reaching R; then, for motion above R, $v = \alpha t = \frac{F}{M} t$, where $F = (m+m'-m)g$, $M = m+m'+m$

$$(\overline{m+m'}+m)v=(\overline{m+m'}-m)gt;$$

$$\therefore v=\frac{m'gt}{2m+m'};$$

$$=\frac{m'g}{2m+m'}; \text{ since } t=1.$$

The velocity now becomes uniform; let RS, which is described in 1″, equal a feet;

$$\text{then } a=v\times 1;$$

$$=\frac{m'g}{2m+m'};$$

$$\therefore g=\frac{2m+m'}{m'}a.$$

If, therefore, RS be measured, g can be found.

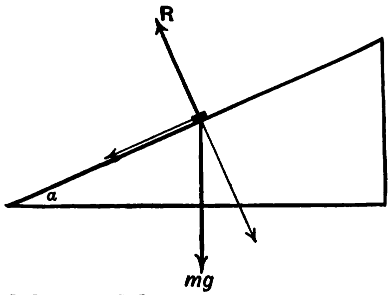
258. The theoretical advantage of the method is that, if m' be small, the velocity will be small; and so a will be small and easily measured. [The time can be measured very accurately by means of a pendulum beating seconds.]

But it is impossible, of course, to get a pulley without mass and without friction; and so we are unable to determine g so accurately by this method as it can be determined by the pendulum.

259. The Inclined Plane.—Yet another method of determining the acceleration of gravity is by means of an inclined plane.

260. The Inclined Plane requires no definition or description. It was used by Galileo for the determination of the value of g, and yields fairly approximate results.

261. If a mass of m lbs. be sliding down a perfectly smooth plane inclined to the horizon at an angle of $a°$; the only forces acting are the attraction of the earth, and the normal pressure of the plane; represented respectively by the vertical mg poundals and the normal R poundals.



The vertical force may be resolved into two, one along the inclined plane and the other at right angles to it.

Clearly, the two components as well as the force are constant in magnitude and direction; so that similar and similarly situated parallelograms of forces maintain throughout.

262. The resolved force at right angles to the plane is $mg \cos a$ poundals. This will exercise no influence at all on the motion (the plane being smooth); and will merely balance the R poundals.

The resolved force along the plane is $mg \sin a$ poundals.

The mass then moves simply under the action of this force; therefore, if l feet be the length passed in t'',

$$ml = \frac{1}{2} mg \sin \alpha \times t^2;$$

$$\therefore l = \frac{1}{2} gt^2 \sin \alpha.$$

263. We see then that, by taking α small, we may make the length passed in $1''$ small, and do away with the difficulty mentioned at the beginning of this section.

But there are obvious practical difficulties in this method; *e.g.* the obtaining of a smooth plane and the exact determination of α.

264. DEFINITION.—A plane is said to have an *Incline of 1 in l*, when for every *l feet of slope* the plane rises *1 foot vertically*.

265. Friction.—The inclined plane is also useful in determining the force of friction between bodies.

266. Machines, wheel carriages, and other moving bodies, are more or less retarded by friction.

No surfaces are perfectly smooth. In all cases there is, to a less or greater degree, an unevenness; and therefore when two surfaces come together the prominent parts of one fall into the hollow parts of the other.

Friction, therefore, acts as a retarding influence in all mechanical contrivances. In many instances it destroys more than half of the power employed.

267. The Laws of Friction between bodies, as found by experiment, are surprisingly simple.

268. Except in extreme cases the following is the only *positive* law.

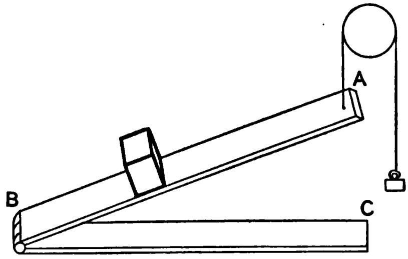
When the materials of the surfaces in contact remain the same, the Friction varies as the Normal Pressure.

269. *Negative* laws are:

The Friction is (except under certain circumstances) quite Independent of the Velocity, and of the extent of Surfaces in contact.

270. Rather more Friction can be called into play when the Friction is Statical than when it is Kinetic. The Friction, *i.e.* between moving bodies, though still proportional to the normal pressure, is less than the Friction which maintains just before the bodies begin to move. But, when the bodies are once in motion, the Friction remains constant.

271. So if a body be resting on a rough board, as the



angle *ABC* increases, more and more friction is called into play up to the moment when the body begins to move.

Directly it begins to move the friction diminishes, and then (speaking generally) remains constant for all velocities.

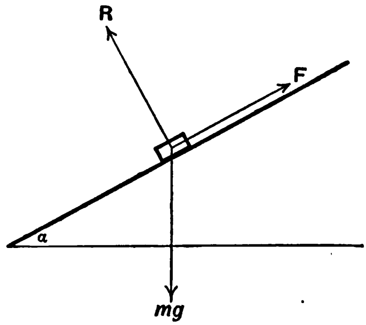
272. It follows that to find the full Statical force of friction between bodies, it is only necessary to place one body on the inclined surface of the other in the position in which the former is only just at rest.

Let $a°$ be the inclination of the surface to the horizon;

m lbs. the mass of the other body;

F poundals the force of friction;

R poundals the pressure of the plane.



Resolving all the forces along and at right angles to the plane,

$$\text{we have } mg \sin \alpha - F;$$
$$\text{and } R - mg \cos \alpha.$$

These must each be equal to zero, since there is no motion;

$$\text{whence } F = R \tan \alpha;$$
$$\text{or if } \tan \alpha = \mu,$$
$$F = \mu R.$$

273. μ is called the *coefficient of friction*; and, as stated above, is constant for the same surfaces.

Also, as stated above, μ will be somewhat less than tan a when the bodies are in relative motion.

WORKED EXAMPLES.

A. Find the number of oscillations gained or lost in a given time when a pendulum is taken from a place where the acceleration of gravity is g sfas to one where it is g' sfas.

If n, n' be the numbers of oscillations at the two places in T seconds;

$$\text{then } \frac{T}{n}=\pi\sqrt{\frac{l}{g}}\,;\ \text{ and } \frac{T}{n'}=\pi\sqrt{\frac{l}{g'}}\,;$$

$$\text{whence } n-n'=\frac{T}{\pi\sqrt{l}}(\sqrt{g}-\sqrt{g'}),$$

$$=n\frac{\sqrt{g}-\sqrt{g'}}{\sqrt{g}}.$$

[A useful approximate value for $n-n'$ is obtained by noting that over the surface of the earth g and g' do not greatly differ, and that therefore approximately

$n=\frac{T}{\pi}\sqrt{\frac{g}{l}}$

$dn=\frac{T}{\pi\sqrt{l}}\frac{dg}{2\sqrt{g}}$

$\frac{dn}{n}=\frac{dg}{2g}$

$$2\sqrt{g}=\sqrt{g}+\sqrt{g'}\,;$$

$$\therefore\ n-n'=n\frac{\sqrt{g}-\sqrt{g'}}{\sqrt{g}}\times\frac{\sqrt{g}+\sqrt{g'}}{2\sqrt{g}}\,;$$

$$=n\frac{g-g'}{2g}.]$$

B. A pendulum which beats seconds at one place gains 2 beats in every 1000 seconds at another place; compare the weights at the two places of the same mass.

$$1=\pi\sqrt{\frac{l}{g}}\,;\ \text{ and } \frac{1000}{1002}=\pi\sqrt{\frac{l}{g'}}\,;$$

$$\therefore\ mg : mg' :: (1000)^2 : (1002)^2$$

$$:: 1 : 1{\cdot}004004.$$

C. To a hook in a ceiling attach two long fine strings of equal length, to the ends of which attach kitchen "weights" of any size. Draw back the "weights" to a small distance in opposite directions (say one of them 2 feet, the other 6 ins.). Show that when let go the theoretical and practical results (as to the time of falling) coincide.

They reach their lowest points at the same instant.

D. How many vibrations will a pendulum 1 foot long make in a minute, at a place where the seconds' pendulum is 38·88 inches?

$$\frac{60}{x}=\pi\sqrt{\frac{1}{g}}; \text{ and } 1=\pi\sqrt{\frac{(38{\cdot}88)\div 12}{g}};$$

$$\text{whence } x=108.$$

E. Explain how to use Atwood's machine to show that a mass acted on by the force of the earth's attraction moves with uniform acceleration.

The velocity after passing the ring is measured, of course, by the distance passed over in 1″;
now this distance is found, by the Machine, to be proportional to the *time* of motion before reaching the ring;
therefore the velocity below the ring is also proportional to the time above;
this shows that the acceleration must have been constant.

F. In an Atwood's machine the equal masses were 1 lb., and the added mass $\frac{1}{5}$ oz.; and the system moved through $2\frac{1}{2}$ feet

before passing the ring. After passing the ring one foot was covered in 1 second. Find the acceleration of gravity.

$$\frac{1}{2}\left(2+\frac{1}{80}\right)v^2=\left(1\frac{1}{80}-1\right)g\times\frac{5}{2};$$

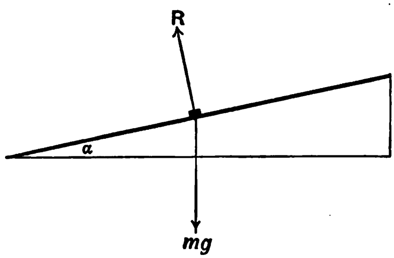
$$\therefore\ v^2=\frac{5}{161}g;$$

whence, from the second part of the motion,

$$1=\sqrt{\frac{5g}{161}}\times 1;$$

whence acceleration of gravity is 32·2 sfas.

G. If a gun recoil with a velocity of 420 cas, and move up a smooth inclined plane rising 1 in 5 to a distance of $4\frac{1}{2}$ metres; find the acceleration of gravity.



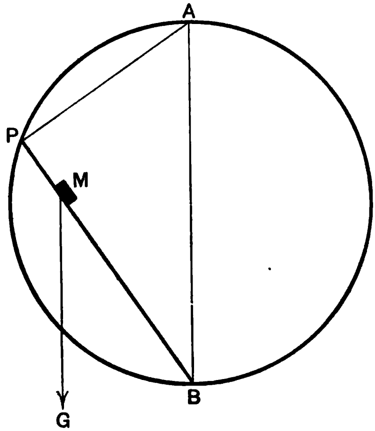
The acceleration of gravity on the inclined plane

$$=g\cos(90^\circ-\alpha^\circ);$$
$$=g\sin\alpha^\circ;$$
$$=\frac{g}{5};$$

$$\therefore\ \frac{1}{2}mv^2=m\frac{g}{5}x;$$

$$\text{whence } g=\frac{(420)^2\times 5}{2\times 450};$$
$$=980.$$

H. Show that the time of falling down a smooth chord of a vertical circle is the same for all chords passing through the lowest point of the circle.



Let AB be the vertical diameter.

The acceleration

$$=g\cos GMB;$$

$$=g\cos PBA;$$

also $PB = AB \times \cos PBA$;

$\therefore$ time is given by

$$\sqrt{\frac{2\,AB\cos PBA}{g\cos PBA}};$$

and is therefore the same for all chords through B.

I. Find the H. P. of an engine required to drag a train of 97 tons up an incline of 1 in 56 with a velocity of 24 miles an hour; when the friction is equal to the weight of half a ton.

The acceleration of gravity is $\frac{g}{56}$ sfas;

so the force required, in poundals,

$$=97\times 2240\times\frac{g}{56}+1120g;$$

$$=5000g;$$

$$\therefore \text{ no. of H.P.}=\frac{5000g\times 24\times 88}{33000g};$$

$$=320.$$

EXAMPLES.—XI.

1. A simple pendulum of length l centimetres swings, at a place where the acceleration of gravity is g scas, through an arc of $2a°$; find the velocity when the pendulum rod is vertical.

2. In the latitude of London the length of the seconds' pendulum is 39·1386 inches; what, to two decimal places, is the value of g? [Take $\pi^2 = 9{\cdot}87$.]

3. Tie a kitchen "weight" at the end of a thin string, so that the distance between the point of suspension and the centre of the mass is 8 feet; and let it oscillate for 11 seconds. Find (theoretically and practically) the number of oscillations. [$g=32$, $\pi=\frac{22}{7}$.]

4. If a seconds' pendulum were a metre in length, what would be the length of a pendulum which would oscillate 25 times in a minute?

5. Use Atwood's machine to show that the weight of a mass at a given place is independent of the velocity with which it is moving.

6. At a place where the acceleration of gravity is 980 scas, the equal masses of an Atwood's machine are each 1 kilogramme. One is placed at 90 cms. above the ring; and the ring is 28 cms. above the stage. What mass must be added to make the system pass from ring to stage in 1 second?

7. A particle slides down $1\frac{4}{5}$ metre of a smooth plane inclined to the horizon at an angle of 30°, and attains a velocity of 420 cas; find the acceleration of gravity.

8. Show that the time of falling from the highest point of

a vertical circle (radius a feet) down any smooth chord, is equal to that of falling down the diameter; and find the length of the chord on falling down which a particle would acquire half the velocity it would acquire in falling down the diameter.

9. A mass of 5 lbs. falls the whole length (130 feet) of a rough inclined plane, whose height is 50 feet and base 120 feet; and acquires 6422 faspen. The retardation due to friction being one-fifth of the acceleration down the plane due to gravity; find the acceleration of gravity.

---

10. Show that the velocity at the lowest point of a simple pendulum varies as the chord of the arc through which it has fallen. (Chd AB)^2 = 2l·AM

11. A pendulum is a seconds' pendulum at a place where the acceleration of gravity is 982·8225 scas. How many beats will it lose in 31 hrs. 21 min. at a place where the acceleration of gravity is 979·69 scas?

12. Two pieces of putty, massing 1 lb. and 2 lbs., are suspended from the same point by long fine strings of equal length. They are drawn back respectively 2 feet and 1 foot in opposite directions, and let go at the same moment. Show that the theoretical and practical results (as to time and subsequent velocity) coincide.

13. At a place where the acceleration of gravity is π^2 scas, find the length of a pendulum which oscillates as often in 1000 seconds as there are centimetres in its length.

14. If, in Atwood's machine, the system move for τ seconds before passing the ring; and take t seconds to pass from the ring to the stage; find the value of g, the masses being m and m' lbs., and the distance from the ring to the stage a feet.

15. In an Atwood's machine the masses are $\frac{7}{8}$ kilogramme and 10 grammes; the distance between ring and stage is 35 cms.; and this is passed in 1 second. The acceleration of gravity being 980 scas, find the distance moved before reaching the ring.

16. If the shot, etc. (massing 322 lbs.) leave a $12\frac{1}{2}$ ton gun with a velocity of 1400 fas; and the gun recoil up a smooth inclined plane, rising 1 in 4, to a distance of 16·1 feet; find the acceleration of gravity.

17. If two vertical circles touch each other at their highest points; and a chord be drawn through the point of contact; the time of falling from rest down the portion of the chord intercepted between the two circles is constant.

18. The surface of a plane inclined at angle of 15° to the horizon is smooth at the upper end for $\sqrt{3}$ yards, and rough at the lower for 6 feet. If a particle starting at the top just come to rest at the bottom, find the coefficient of friction.

---

19. The velocity at any point of a simple pendulum depends simply on the position of the point relatively to the highest point, no matter the amount of swing.

20. At a place where the acceleration of gravity is 980 scas, in what time would a pendulum oscillate whose length is one-fifth of a metre? [$\pi = 3\frac{1}{7}$].

21. Show, theoretically and practically, that on the surface of the earth the length of the seconds pendulum is about a metre.

22. Assuming that the force of gravity varies inversely as the square of the distance from the centre of the earth; and that the radius of the earth is 4000 miles; find the time of oscillation of a simple pendulum 144 cms. long, at a place 2 miles above a spot

on the earth's surface, where the seconds' pendulum is assumed to be a metre.

23. Use Atwood's machine to show that under the same circumstances the momentum is constant whatever the mass.

24. If in an Atwood's machine the masses are 539 and 12 grammes; and if the system move through $\frac{3}{8}$ metre before passing the ring, and through 36 cms. in 1″ after passing the ring; find the acceleration of gravity.

25. A smooth plane, inclined to the horizon at 45°, is made to move on a smooth horizontal plane, with such an acceleration that a particle on the inclined plane remains at rest. Find the acceleration.

26. If P be a point on the circumference of a vertical circle (radius a feet), whose highest and lowest points are A and B respectively; and if TP, TB be tangents; find the time of falling down PT in terms of the chord AP.

27. At what angle must the least force act which will drag a mass along a rough horizontal plane; the friction being such as would just prevent the body from sliding down a plane rising 1 in 2?

---

28. Assuming that the force of gravity varies inversely as the square of the distance from the centre of the earth; find the time of a small oscillation of a seconds' pendulum, in a balloon h miles high, the diameter of the earth being $2R$ miles.

29. The earth's radius being 4000 miles, and the attraction of gravity being taken as varying inversely as the square of the distance from the centre of the earth; if a pendulum gain $\frac{1}{4}$″ in 1000″ at the surface; at what height will it keep good time?

30. Two pendulums, the longer of which is a metre, begin to oscillate together, and are again together after 20 oscillations of the shorter. Find the length of the shorter, and test practically.

31. At a certain place a pendulum a yard long was set up as a seconds' pendulum, but was found to lose 6 seconds in the hour; by how much must it be shortened?

32. If in Atwood's machine (masses m and m' lbs.) the system move through l feet before passing the ring, and through a feet in t'' after passing the ring; find the value of g.

33. In an Atwood's machine the times before and after passing the ring being each $5''$; and the masses being $2\frac{1}{4}$ lbs. and 1 oz.; find the distance between the ring and stage; the acceleration of gravity being 32·12 sfas.

34. Find the pressure on a smooth plane, with an inclination of 1 in 25·01, when a mass of 125 tons 1 cwt. is just kept in equilibrium upon it by a force which acts in a direction making an angle of 45° with the inclined plane.

35. Find the line of shortest descent from a given point P to a given straight line AB.

36. A mass of 500 tons 1 cwt. is to be dragged up an incline of 1 in 50·005 at the rate of 20 miles an hour. The resistance to motion (owing to friction) being 10 lbs. weight per ton; find the H.P. required, and the pressure on the plane.

---

37. Show that the velocity at the lowest point of a simple pendulum varies as the square root of the versed sine of the arc through which it has fallen.

38. The acceleration of gravity within the earth being assumed[1] to vary directly as the distance from the centre; if at the surface, and at 50·3 kilometres within the surface, a pendulum makes respectively 252 and 251 oscillations in a given time; find the radius of the earth supposed spherical.

39. Two unequal pieces of putty are suspended from the same point by long fine strings of equal length. They are drawn back equal distances in opposite directions and simultaneously let go. What is the theoretical result? Test it practically.

40. Taking the earth's radius as $6375\frac{5}{8}$ kilometres, and assuming[1] that down a mine the value of g varies directly as the distance from the centre of the earth; find how many beats will be lost by a seconds' pendulum when taken to a depth of 100,996 centimetres.

41. Explain how to use Atwood's machine to show that the momentum of a given mass after a given time is proportional to the force which acts upon it.

42. In an Atwood's machine (masses $2\frac{1}{2}$ lbs. and $\frac{1}{2}$ oz.) it was found that the system passed from the ring to the stage, a distance of 2 feet, in $2\frac{1}{2}''$. The acceleration of gravity being 32·2 sfas; for how long had the system been in motion before reaching the ring?

43. Find the H.P. of an engine required to drag a train of 75 tons up a smooth incline of 1 in 35, at the rate of 25 miles an hour.

1 This is only an assumption. As a fact g increases for some way into the earth.

44. If the highest point on a vertical circle be joined to a point outside the circle, show that the portion of the line intercepted between the circle and the point is the line of shortest descent from the circle to the point.

45. If a mass be dragged up a rough inclined plane, show that the work done is equivalent to the work done in dragging it along the horizontal base of the inclined plane and up the vertical height; the coefficient of friction being the same on the inclined and horizontal planes.

SECTION XII.

APPLICATIONS OF THE LAWS.

A.—*Collision.*

274. If a glass ball be dropped on a slab of polished marble it will rebound almost to the point from which it is let fall.

But if a cricket-ball be dropped, though it will also rebound, it will not rise to anything like the same height.

Technically the one is more elastic than the other.

275. If, when the ball is dropped, the polished slab on which it falls be slightly smeared with oil, the ball will be found to have upon it a circular drop of oil larger or smaller according to the height of the fall.

So when the ball is dropped it is flattened, and rebounds owing to the effort to recover its shape.

276. As the result of many experiments the following is found to be the law of elasticity.

Law of Elasticity.—**When two spheres of the same substance collide, their relative velocity after impact always bears a constant relation to their relative velocity before impact.**

This ratio is called the Elasticity; and is usually denoted by the symbol e.

277. The general problem of **direct collision** is this:

Two balls of masses m and m' grammes are moving, with velocities u and u' cas, in the same direction; u being greater than u', the balls come into collision; find the velocities (v and v' fas) after the impact (gravity and the resistance of the air not being considered).

From the law of elasticity we get,

$$v'-v=e\,(u-u');$$

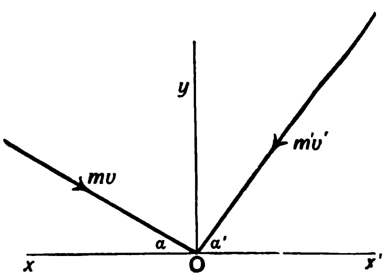
and, as there is no external force, the total momentum must (by the 1st law of motion) remain constant;

$$\therefore\ m'v'+mv=mu+m'u'.$$

These two equations are sufficient to determine v and v'.

278. In cases of **oblique collision**, it is only necessary to resolve the velocities in two directions, one being *in the line joining the centres* of the balls *at the moment* of collision, the other *at right angles* to this through their point of contact.

279. Thus supposing two smooth balls, of masses m and m' lbs., moving (as in the figure) with v and v' fas respectively, to collide so that their line of centres is xOx'; with which line their directions make angles respectively of a and a'.



Then we have a direct collision of m and m' lbs., moving with $v \cos a$ and $-v' \cos a'$ fas respectively;

$$\therefore \; V' \cos \beta' - V \cos \beta = e\,(v \cos a + v' \cos a');$$

$$\text{and } m'V' \cos \beta' + mV \cos \beta = mv \cos a - m'v' \cos a';$$

[V and V' fas, β and β', being the velocities and directions after collision].

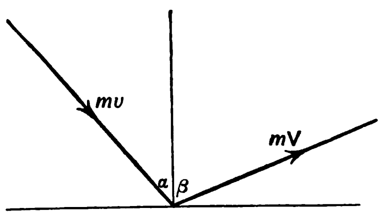
Also as there is nothing to disturb the velocities in the direction yO, the balls being smooth,

$$V \sin \beta = v \sin a;$$

$$V' \sin \beta' = v' \sin a';$$

these four equations are sufficient to determine V, V', β and β'.

280. In the case of a ball impinging **obliquely on a smooth plane.**



If a and β degrees be the angles of incidence and reflection;

m lbs. the mass of the ball;

v and V fas the velocities before and after impact; as the plane is smooth, and there is consequently no force in that direction, the momentum in the direction of the plane is constant;

$$\therefore \; mV \sin \beta = mv \sin a;$$

Perpendicularly to the plane, as the plane has no velocity,

the velocity after must be e times, the velocity before collision;

$$\therefore \; V \cos \beta = ev \cos \alpha;$$

from which two equations V and β can be found.

281. We may notice that

$$\cot \beta = e \cot \alpha.$$

WORKED EXAMPLES.

A. Two spherical balls A and B are suspended by strings, so that when at rest the balls just touch, their centres being in the same horizontal line. A is drawn back to a vertical height of a feet (the string being kept taut) and let go. By the collision, B is driven to a vertical height of b feet; and A rebounds to a height of c feet. Show that $(\sqrt{b} + \sqrt{c}) \div \sqrt{a}$ is constant for any masses of the same substance and for all values of a.

The velocities before collision are $\sqrt{2ga}$ fas and O; the velocities after collision, $-\sqrt{2gc}$ fas, and $\sqrt{2gb}$ fas;

$\therefore$ by the law of elasticity,

$$\sqrt{2gb} + \sqrt{2gc} = e(\sqrt{2ga});$$

$$\text{or } (\sqrt{b} + \sqrt{c}) \div \sqrt{a} \text{ is constant.}$$

[illegible] fr. art. 277

This indicates a method by which the Law of Elasticity is found. $\sqrt{b} + \sqrt{c}$ being found, in all experiments, to bear a constant ratio to $\sqrt{a}$, it follows that the difference of the velocities after bears a constant ratio to that before the collision.

B. Two equal balls are moving in the same direction with velocities one double of the other. They collide and lose by

the impact one-eighteenth of their kinetic energy. Find e and the ratio of their velocities after impact.

If $2v$ and v fas be their initial velocities;

V' and V fas their final velocities;

the momentum equation gives

$$V+V'=3v;$$

and the law of elasticity gives

$$V-V'=ev;$$

whence $$V=\frac{3+e}{2}v; \quad V'=\frac{3-e}{2}v;$$

$$\therefore \left(\frac{3+e}{2}\right)^2+\left(\frac{3-e}{2}\right)^2=\frac{17}{18}(1+4);$$

$$\therefore e=\frac{2}{3};$$

and required ratio is 7 : 11.

C. If a ball impinge on a smooth plane in a direction making an angle of 30° with the normal at the point of impact; find the coefficient of elasticity that the direction of reflection may make an angle of 60° with the normal; and find the relation between the kinetic energies before and after impact.

Referring to the figure in the text

we have $V\sin 60°=v\sin 30°$;

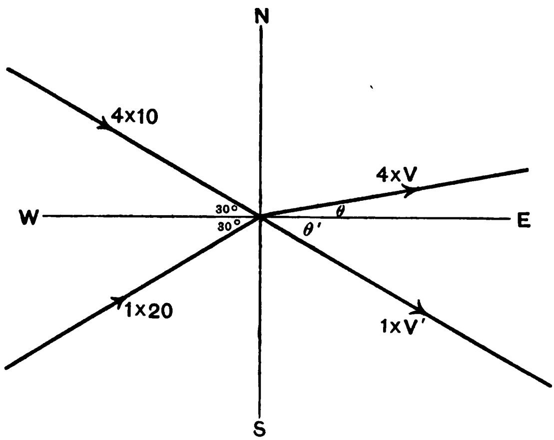
and $V\cos 60°=ev\cos 30°$;

whence $e=\frac{1}{3}$;

$$\text{and } \frac{\frac{1}{2}mv^2}{\frac{1}{2}mV^2}=\frac{\sin^2 60°}{\sin^2 30°};$$
$$=3.$$

D. Two smooth balls, of elasticity $\frac{2}{3}$ and of masses 4 lbs. and

1 lb., are moving with velocities of 10 and 20 fas in directions the one 30° to the north of E., the other 30° to the south of E. They collide so that the line joining their centres is due N. Find their velocities and directions after collision.



The balls being smooth there is no change of velocity on the line *WE*;

$$\therefore \; V\cos\theta = 10\cos 30° = 5\sqrt{3};$$

$$\text{and } V'\cos\theta' = 20\cos 30° = 10\sqrt{3}.$$

For the momentum equation on the line *NS*, we have

$$V'\sin\theta' - 4V\sin\theta = 40\cos 60° - 20\cos 60° = 10;$$

and for the elasticity equation on the same line,

$$V'\sin\theta' + V\sin\theta = \frac{2}{3}(10\cos 60° + 20\cos 60°) = 10.$$

From these four equations we easily find that the upper ball travels due E. with $5\sqrt{3}$ fas; the lower with 20 fas in a direction 30° to the south of E.

EXAMPLES.—XII.

1. When two balls moving in the same direction collide, show that the momentum lost by the hinder ball at the moment of greatest compression varies as the difference of the velocities and as the harmonic mean between the masses.

2. Two balls of mass 5 lbs. and 6 lbs. are moving in opposite directions with velocities respectively of 55 and 44 fas. If the coefficient of elasticity be $\frac{2}{3}$; find the velocities after impact.

3. At what inclination to the normal must a ball (elasticity $\frac{1}{3}$) be incident on a smooth hard plane that it may rebound in a direction perpendicular to that in which it strikes?

What is the greatest inclination to the normal for which it is possible for these two directions to be at right angles?

4. An inelastic ball of mass 3 lbs., moving with a velocity of 13 fas, impinges upon a second inelastic ball of mass 10 lbs. at rest; its line of motion making an angle 30° with the line joining the centres of the balls at the moment of collision. Find the subsequent velocity of the first ball.

---

5. Show that a perfectly inelastic ball will, after impact with a smooth plane, slide along the plane.

6. Show that if equal masses collide, the gain of velocity by one is equal to the loss by the other.

7. A ball impinges on a smooth plane in a direction making an angle of 30° with the normal, and loses half its kinetic energy; find the elasticity and angle of reflection.

8. A ball of mass m lbs. impinges on one of m' lbs. at rest, its line of motion making an angle of $a°$ with the line joining the

centres of the balls. If the new line of motion of m is at right angles to the old, find the elasticity.

---

9. When two balls (m and m' lbs.) are moving in the same direction, find the elasticity that the velocities of the swifter balls before and after collision may be the same.

10. Three perfectly elastic balls, of 6 lbs., 3 lbs., and 1 lb., are in a row. The 1st is made to move with a velocity of a yard a second; find the momentum and the kinetic energy of each after impact.

11. A ball of elasticity $\frac{1}{2}$ is projected from a point P on the circumference of a smooth circle; and after two reflections returns to P. Find the tangent of the angle which the direction of projection makes with the radius through P.

12. Two equal smooth balls, of elasticity $\frac{1}{4}$, are moving, in directions inclined at an angle of 60°, with equal velocities 16 fas; and impinge so that the line joining the centres at the moment of impact, is in the direction of the motion of the hinder ball. Find the velocities after collision.

---

13. A ball impinges on one smooth plane and then on another at right angles to the first. All the motion being in one plane, and e being the same in both cases, show that the initial and final directions are parallel.

14. A perfectly elastic ball of m lbs. impinges on another of m' lbs. at rest; find the condition that after collision the kinetic energies may be equal. Explain the double answer.

15. Show that a billiard ball of any elasticity, striking against the sides of the table in order, describes a parallelogram.

16. Two billiard balls are lying in contact, and a third moving with v fas strikes them both at the same moment. The balls being perfectly smooth and elastic, find the subsequent velocities.

---

17. Show that the difference of the total kinetic energy before and after impact of two balls, of given elasticity, moving in the same direction, varies as the square of the difference of the velocities, and as the harmonic mean between the masses. In what case is there no loss of kinetic energy?

18. An inelastic mass of m grammes impinges on another of m' grammes at rest; in what ratio is the total kinetic energy diminished?

19. If the ball of qu. 15 return after striking all four sides to the point it started from; show that, the elasticity being perfect, the sides of the parallelogram are parallel to the diagonals of the table.

20. A smooth ball, of elasticity $\frac{2}{3}$ and mass 1 lb., moving at the rate of 30 fas, impinges *directly* on a similar ball, of mass 4 lbs., moving in a direction at right angles to that of the first ball. Find the decrease in momentum of the 1st ball; and the increase of K. E. in the 2nd. In what direction will the 1st ball move after collision? If the 2nd ball after collision moves in a direction inclined at an angle of 45° to its former direction, what must have been its velocity?

---

21. If a ball of mass m lbs. impinge on one of m' lbs. at rest, show that, if e be not less than $\frac{2\sqrt{mm'}}{m+m'}$, a ball can be placed between m and m', so that the velocity communicated to m' through

this third ball may be equal to that communicated direct; and find the mass when $e=\dfrac{2\sqrt{mm'}}{m+m'}$.

22. Four perfectly elastic balls are placed in a row, not touching; and an extreme one is set in motion. Find the ratio of the masses when, after impact, all have the same momentum.

23. A billiard ball P is placed so that, AB being one side of the table, APB is a right angle. Another ball Q is sent in a direction QP, such that $QPB=QPA=135°$. If after impact P goes into the pocket at A; with what velocity and in what direction will Q move; (both balls being perfectly elastic and smooth)?

24. Two equal perfectly elastic billiard balls are lying in contact on a table, when a third, equal to either, strikes one of them obliquely. Determine practically and theoretically the directions in which they will move off.

SECTION XIII.

APPLICATIONS OF THE LAWS.

B.—*Projectiles.*

282. Definition.—A Projectile is a mass which, being set in motion, moves under the action of gravity.

283. It is obvious that the action of the air will, in all projectiles, largely affect the results. To introduce, however, the correction for the air-resistance requires the processes of the higher mathematics. It is interesting to obtain such results as we can from the simple problem.

284. The case in which the projectile is fired vertically presents no difficulty.

285. The case in which a mass is projected horizontally has been considered as an illustration of the 1st and 3rd laws of motion.

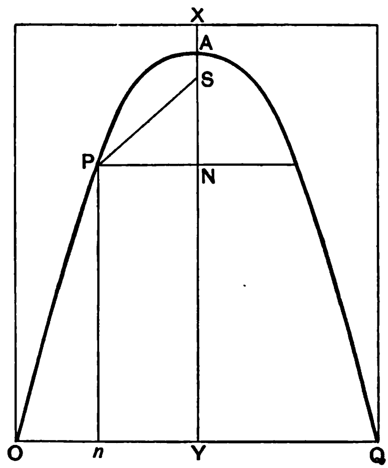
286. Take the general case. A mass, m lbs., is projected from O with a velocity of V fas, in a direction making an angle a° with the horizon (the resistance of the air being neglected); find the curve described, and the circumstances of the motion.

Let P be the position at the end of t'';

v fas the then velocity;

θ° the inclination of the direction to the horizon;

X and Y feet the lengths of On and nP.



By the principle of the resolution of velocities we may consider the mass to have independent horizontal and vertical velocities

initially of $V \cos \alpha$ and $V \sin \alpha$ fas;
at the end of t'' of $v \cos \theta$ and $v \sin \theta$ fas.

There is nothing to interfere with the horizontal velocity,

$$\therefore v \cos \theta = V \cos \alpha; \quad \ldots \ldots \ldots \quad (1)$$

but the vertical velocity is diminished by the acceleration of gravity,

$$\therefore v \sin \theta = V \sin \alpha - gt; \quad \ldots \ldots \ldots \quad (2)$$

and we have further

$$X = Vt \cos \alpha\, ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$Y = Vt \sin \alpha - \frac{1}{2} g t^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These four equations determine the position and velocity (*i.e.* all the circumstances of motion) at any time.

287. By eliminating t between equations (3) and (4), we obtain Y in terms of X;

$$Y = X \tan \alpha - \frac{g}{2V^2 \cos^2 \alpha} X^2\, ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which students of analytical conics will recognise as the equation to a parabola whose

$$\textbf{Latus Rectum} = \frac{2}{g}(V \cos \alpha)^2 \textbf{ feet}. \quad . \quad . \quad . \quad . \quad (6)$$

288. From (4) the ordinate Y will vanish

$$\text{if } Vt \sin \alpha - \frac{gt^2}{2} = 0;$$

i.e. when $t = 0$; and when $t = \dfrac{2V \sin \alpha}{g}$;

$t = 0$, of course, when the mass starts;

$t = \dfrac{2V \sin \alpha}{g}$ gives the time of arrival at the point Q, where the mass strikes the horizontal plane through O;

$$\therefore \textbf{ Whole time of flight} = \frac{2V \sin \alpha}{g} \textbf{ seconds}. \quad . \quad . \quad . \quad . \quad (7)$$

289. OQ is called the range; and from (3) we get

$$\textbf{Range} = \frac{V^2 \sin 2\alpha}{g} \textbf{ feet}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

290. P will be at its highest position when Y is greatest; *i.e.* from (4), when

$$Vt \sin \alpha - \frac{1}{2} gt^2 \text{ is greatest;}$$

$$i.e. \text{ when } \frac{V^2 \sin^2 \alpha}{2g} - \frac{1}{2} g \left(\frac{V \sin \alpha}{g} - t \right)^2 \text{ is greatest;}$$

$$i.e. \text{ when } \frac{V \sin \alpha}{g} - t = o;$$

$\therefore$ **time to highest point = half time of flight**; . . . (9)
as is otherwise evident.

291. From equations (4) and (9)

$$\textbf{Greatest Height} = \frac{V^2 \sin^2 \alpha}{2g} \textbf{ feet.} \quad . \quad . \quad . \quad (10)$$

292. From equations (3) and (9), or otherwise,

$$OY = \textbf{half the Range.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

293. If now $AN = x$ feet, and $NP = y$ feet, we obtain

$$\text{from (11) and (3) } y = \frac{V^2 \sin 2\alpha}{2g} - Vt \cos \alpha; \quad . \quad (12)$$

$$\text{and from (10) and (4) } x = \frac{1}{2g} (V \sin \alpha - gt)^2; \quad . \quad . \quad . \quad (13)$$

$$\therefore \ y^2 = V^2 \cos^2 \alpha \left(\frac{V \sin \alpha}{g} - t \right)^2;$$

$$= \frac{2 (V \cos \alpha)^2}{g} x; \quad . \quad . \quad . \quad . \quad (14)$$

$$\text{or } PN^2 = 4\, AS \times AN;$$

if AS be taken equal to $\frac{(V \cos \alpha)^2}{2g}$ feet. . . . . . . (15)

As before, the Latus Rectum $= \frac{2 (V \cos \alpha)^2}{g}$ feet;
where $V \cos \alpha$ fas is the constant horizontal velocity.

WORKED EXAMPLES.

A. From equation (5) determine the Range and the Greatest Height; and show that the curve is symmetrical about AY.

For Range put $Y=0$,

$$\therefore \text{ Range}=\frac{2V^2\cos^2\alpha\tan\alpha}{g}\text{ feet};$$

$$=\frac{V^2\sin 2\alpha}{g}\text{ feet.}$$

For Greatest Height

$$X=\text{Half Range};$$

$$=\frac{V^2\cos^2\alpha\tan\alpha}{g}\text{ feet};$$

$$\therefore \text{ Greatest Height}=\frac{V^2\sin^2\alpha}{2g}\text{ feet.}$$

To show symmetry it is only necessary to show that Y is the same

when $X=Z$,

and when $X=\text{Range}-Z$;

and it is easy to throw equation (5) into the form

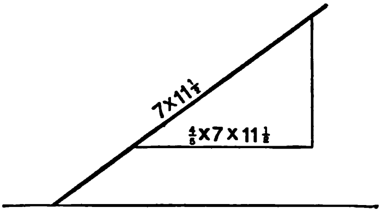
$$\frac{2Y(V\cos\alpha)^2}{g}=\left(\frac{V^2\sin\alpha\cos\alpha}{g}\right)-\left(\frac{1}{2}\text{ Range}-X\right)^2;$$

and as $\left\{\frac{1}{2}\text{Range}-(\text{Range}-Z)\right\}=\left\{\frac{1}{2}\text{Range}-Z\right\}^2$;

$\therefore$ we get the same value for Y in each case.

B. At a place where the acceleration of gravity is 32·2 sfas, a piece of mortar is flicked horizontally from the top round of a ladder and strikes the 8th round. The rounds being $11\frac{1}{2}$ inches

apart, and the inclination of the ladder to the horizon $\tan^{-1}\frac{3}{4}$; find the velocity of projection.



If V fas be the horizontal velocity; and t'' the time;

$$Vt = \frac{4}{5}\times 7\times 11\frac{1}{2}\times\frac{1}{12};$$

$$\frac{1}{2}gt^2 = \frac{3}{5}\times 7\times 11\frac{1}{2}\times\frac{1}{12};$$

whence velocity $= 10\cdot7\dot{3}$ fas.

C. Give a direct proof that a mass projected in a vacuum in any non-vertical direction will describe a parabola.

Let $OT (= V\times t$ feet) be the distance the mass would have described in t'' had there been no gravity;

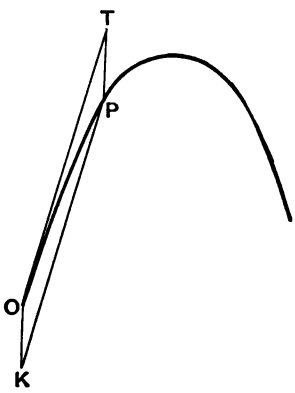
and let TP $(=\frac{1}{2}gt^2$ feet) be the distance down which the body will fall in t'' owing to gravity;

then P is the actual position at the end of t'';

complete the parallelogram $OKPT$;

$$\text{then } PK^2 = V^2t^2;$$

$$= \frac{2V^2}{g}OK;$$



which shows the path to be a parabola.

D. If at a place where the acceleration of gravity is 980 scas, a mass be projected with a velocity of $\frac{3920}{\sqrt{3}}$ cas in a direction making an angle of 60° with the horizon; find the time of flight; the greatest height reached; and the height of the directrix and focus.

Referring to the figure in the text

$$X = V\cos 60° \times t = \frac{1960}{\sqrt{3}}t;$$

$$Y = V\sin 60° \times t - \frac{1}{2}gt^2 = 1960t - 490t^2;$$

$$\therefore \text{ Time of Flight} = \frac{1960}{490} = 4 \text{ seconds};$$

$$\therefore \text{ time to highest point} = 2'';$$

$$\therefore \text{ Greatest Height} = 2V\sin 60° - 2g;$$

$$= 1960 \text{ centimetres};$$

$$\text{Height of Directrix} = \frac{V^2}{2g};$$

$$= 26{\cdot}1\dot{3} \text{ metres};$$

$$\text{Height of Focus} = 26{\cdot}1\dot{3} - 2(26{\cdot}1\dot{3} - 19{\cdot}6);$$

$$= 13{\cdot}0\dot{6} \text{ metres}.$$

Examples.—XIII.

1. From the 3rd law of motion and from equation (4), find the velocity after t''; and show that the expression coincides with that derived from equations (1) and (2).

2. A marble of one ounce mass, being thrown horizontally with $5\frac{1}{2}$ faspen of kinetic energy, is found to have moved horizontally through 10 feet while it has fallen vertically through 9 ft. $1\frac{1}{2}$ ins. Find the acceleration of gravity.

3. Find the angle which the line from the point of projection to the focus of the parabola described by a projectile makes with the horizontal.

4. If the initial direction of a projectile in a vacuum is 60°; compare the heights of the focus and directrix.

---

5. Find the tangent of the angle of projection so that the projectile may have ascended vertically through k feet when it has passed horizontally through h feet. What is the least value of V^2 for which this is possible?

6. A particle whose elasticity is $\frac{11}{16}$ is projected horizontally with a velocity of a mile a minute from a point 256 feet above a horizontal plane. Show that the focus of the parabola described after the first rebound lies in the plane; the acceleration of gravity being 32 sfas. [See qu. 3.]

7. Show that the velocity of a projectile is the same at the same height from the ground, whether the mass be rising or falling.

8. A body is projected at an elevation of 5° (at a place where the acceleration of gravity is 32 sfas) with a velocity of 1000 fas; find the range, time of flight, and greatest height. [Take $\sin 5° = \frac{1}{12}$; $\sin 10° = \frac{1}{6}$.]

---

9. Show that if two particles are projected simultaneously non-vertically from the same point they will never meet.

10. Particles are projected horizontally from the same point and in the same vertical plane with different velocities. Show that the extremities of all the Latera Recta lie on a straight line.

11. Given the greatest height to which a projectile rises to be H feet; find the whole time of flight.

12. Compare the ranges and latera recta of two masses projected from the same point with the same velocity, at elevations of 15° and 30°.

---

13. In the case of a projectile show (without assuming the curve to be a parabola) that there is a point whose distance from the projectile is always equal to the distance of the projectile from a fixed horizontal straight line.

14. A ball standing near the edge of the top of a vertical cliff is struck by a ball, of equal mass, moving (horizontally and at right angles to the edge of the cliff) with a velocity of 12 miles an hour. The cliff being 400 feet high, find the distance from it at which the ball will strike the horizontal plane through its base. [Acceleration of gravity$=32$ sfas; and coefficient of elasticity$=\frac{7}{11}$.]

15. Show that for a projectile in a vacuum the greatest range for a given velocity is when the angle of projection is 45°.

16. The range of a projectile being 1200 yards, and the time of flight 15 seconds; find the direction of projection, the greatest height reached, the latus rectum of the parabola described, and the position of the focus. [Acceleration of gravity$=32$ sfas.]

---

17. If a projectile, fired at an angle of $\alpha°$, has risen k feet while it has moved horizontally through h feet; find (in terms of h, k, α) the ratio of the distance it would have moved had

there been no gravity, to the distance it has fallen owing to the action of gravity.

18. The risers and treads of a flight of stairs are respectively 15 cms. and $22\frac{1}{2}$ cms.; and a marble projected horizontally from the top strikes the edge of the 6th tread. Find the velocity of projection; the acceleration of gravity being 980 scas.

19. Particles are projected simultaneously from a point with a given velocity in all directions in a vertical plane; show that after any time, t'', they are all on a circle; and find its radius.

20. A shell is projected at an angle of 45°, and bursts at the end of the range; and its explosion is heard after $2\frac{171}{175}$ seconds. Taking the velocity of sound at 1 kilometre in 3 seconds, and the acceleration of gravity as 980 scas, and neglecting the air-resistance, find the range and the velocity of projection.

SECTION XIV.

APPLICATIONS OF THE LAWS.

C.—*Circular Motion.*

294. ONCE it was a maxim that "circular motion is perfect."

Modern science teaches us that a mass left to itself will move, not in a circle, but in a straight line; and that if it is to move in a circle force is necessary.

295. When a mass[1] moves in a circle, the force which compels it to describe the circle is called the *Centripetal Force.*

296. This centripetal force produces in the mass in a given time *Normal Momentum*; and it is this which, combined with the *Tangential Momentum* it already has, causes the mass to keep on moving in the circle.

297. If the existing tangential momentum of the mass be suddenly destroyed without the destruction of the centri-

[1] More correctly, its centre.

petal force (as might *e.g.* happen if two planets collided) the centripetal force will drag the mass to the centre of the circle.

On the other hand, if the centripetal force suddenly cease (as would *e.g.* happen if a stone be whirled round at the end of a string and the string break[1]) the mass will move off along the line of the tangent to the circle.

298. Some theorems connected with the motion of a mass in a circle are the following.

299. THEOREM A.—*If a mass describe a circle with uniform velocity, it must be acted on by a force tending to the centre of the circle.*

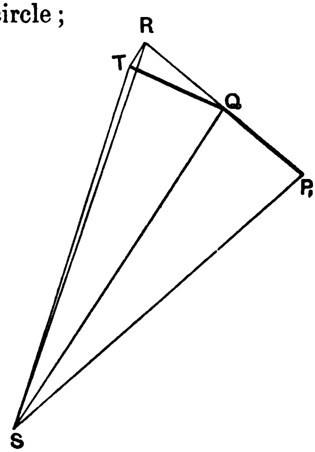
For, the velocity being uniform, there is no alteration of the kinetic energy, and therefore (by the 3rd law) the force, whatever it is, is doing no work.

Hence the force is always at right angles to the direction of the motion; *i.e.* is always directed towards the centre.

300. THEOREM B.—*When a mass of* m *lbs. describes a circle, of radius* r *feet, with uniform velocity* v *fas; to show that the centripetal force is*

$$\frac{mv^2}{r} \textit{ poundals.}$$

[1] A term often used is *Centrifugal Force*; but though the strain of the stone on the string, in the above illustration, might be called a centrifugal force, there seems no sufficient reason for the use of the term.



For S being the centre of the circle;
P_1QT a portion of the circle (see page 74);
t seconds the time of describing P_1Q or QT;
and angle $QSP_1 = a°$;
we clearly have

$$P_1Q = P_1S \times \angle P_1SQ;$$
$$= ra;$$

and, also,

$$P_1Q = vt;$$
$$\therefore ra = vt.$$

Now if f poundals be the force we have (by the 2nd law)

$$ft^* = mv \times \sin RQT;$$

[for the additional momentum produced, which is the equi-

---

* The student should take this opportunity of noticing an important difference in the results obtained from the 2nd and 3rd laws.

In the 3rd law the *direction* of the kinetic energy does not matter. If the velocity is the same in *amount* before and after the action of the force, the kinetic energy is the same.

But, with the 2nd law, a change of *direction* in the velocity indicates a change of momentum even though the velocities before and after may be the same in *amount*.

Thus, in the revolution round the circle, the kinetic energy is to be reckoned as remaining the same, but the momentum changes; and rightly so, for there has been force acting, and force must produce momentum, though it may not produce kinetic energy.

If this appears difficult, the student can but bear it in mind, and test it by subsequent examples. The laws of motion are not self-evident.

valent to ft, is the velocity along QT resolved along the force;]

$$\therefore\ ft = mv \times \sin P_1SQ;$$

$$= mv\alpha;$$

[since α is so small that $\sin\alpha = \alpha$;]

$$= mv \times \frac{vt}{r};$$

$$\therefore\ f = \frac{mv^2}{r}.$$

301. THEOREM C.—*When a mass of* m *grammes describes a circle, of radius* r *centimetres, with a constant velocity in* T *seconds; to find the velocity and the centripetal force.*

Clearly $\qquad vT = 2\pi r;$

$$\therefore\ \text{velocity} = \frac{2\pi r}{T} \text{ cas};$$

$$\text{and } f = \frac{mv^2}{r};$$

$$= \frac{m \times 4\pi^2 r^2}{T^2 r};$$

$$\therefore\ \text{centripetal force} = \frac{4\pi^2 mr}{T^2} \text{ dynes.}$$

302. THEOREM D.—*When masses describe circles, with uniform velocities, round the same centre of force, which is such that it attracts directly as the mass and inversely as the square of the distance; to show that the squares of the periodic times are as the cubes of the radii.*

Let masses of m_1 and m_2 lbs. describe circles, of radii r_1 and r_2 feet, round the same centre of force; and let the times of revolution be T_1 and T_2 seconds; then, if f_1 and f_2 poundals be the respective centripetal forces,

$$f_1 : f_2 :: \frac{m_1}{r_1^2} : \frac{m_2}{r_2^2};$$

$$\text{or } \frac{4\pi^2 m_1 r_1}{T_1^2} : \frac{4\pi^2 m_2 r_2}{T_2^2} :: \frac{m_1}{r_1^2} : \frac{m_2}{r_2^2};$$

$$\text{or } T_1^2 : T_2^2 :: r_1^3 : r_2^3;$$

the same result as that found in paragraph **135**.

WORKED EXAMPLES.

A. If a small can of water be tied to the end of a string and swung round quickly in a vertical circle, the water remains in the can. Why?

Take any particle of the water; then, if the mass-acceleration of the particle caused by the circular motion is greater than its weight, the particle cannot approach the centre of revolution.

B. A mass attached to a string, 3 yards long, revolves in a vertical circle at a place where the acceleration of gravity is $32\frac{1}{9}$ sfas. Find the velocity at the highest point that the string may just remain taut.

This implies that, at the highest point, the weight must just balance the outward mass-acceleration due to the revolution; or, if m lbs. be the mass, we must have

$$mg = \frac{mv^2}{r};$$

whence the velocity = 17 fas.

C. Taking the earth a sphere of radius 4000 miles, and the moon as revolving in $27\frac{1}{2}$ days in a circle round the earth at a distance (centre to centre) of 240,000 miles; and assuming that the earth attracts external bodies with a force varying inversely as the square of the distance from its centre; find the acceleration of gravity at the surface of the earth. $[\pi = 3\frac{1}{7}.]$

If g sfas be the acceleration at the surface, the mass-acceleration of the moon

$$= mg\left(\frac{4000}{240000}\right)^2 \text{ sfasp};$$

$$= \frac{mg}{(60)^2} \text{ sfasp};$$

but the central force (in poundals)

$$= \frac{4\pi^2 mr}{T^2};$$

$$= \frac{4 \times (22)^2 \times m \times 240000 \times 1760 \times 3}{49 \times (27\frac{1}{2} \times 24 \times 60 \times 60)^2};$$

$$= \frac{176m}{49 \times 405};$$

$$\therefore \frac{g}{3600} = \frac{176}{49 \times 405};$$

$$\therefore g = 31\frac{409}{441}.$$

This result is too small; but the earth is not exactly a sphere, the moon does not revolve exactly in a circle, and round numbers only have been taken for the radius, distance, and periodic time. So that the value is as near as could be expected.

It was with this problem that Newton first tested the law of Universal Gravitation; viz., that all masses attract one another with a force varying inversely as the square of the distance.

D. A mass of m lbs. is fastened to one end of a massless string (of length l feet), whose other end is fixed. The mass is then set revolving, in a horizontal circle, so as to make one revolution in T seconds; find the inclination of the string to the vertical; the acceleration of gravity being g sfas.

The forces acting on the mass are its weight and the tension of the string; and the horizontal centripetal force is the resultant of these two.

If Θ be the inclination of the string to the vertical, it is easily seen that the horizontal centripetal force (in poundals)

$$= mg \tan \Theta; \text{ (by the parallelogram of forces)};$$

$$\text{and also} = \frac{4\pi^2 ml \sin \Theta}{T^2}; \text{ (by theorem C)};$$

$$\text{whence } \cos \Theta = \frac{gT^2}{4\pi^2 l}.$$

This apparatus is called a *Conical Pendulum.*

EXAMPLES.—XIV.

1. Why does water, set rotating in a basin, rise at the sides and sink away at the middle?

2. A railway truck is moving at the rate of 45 kilometres an hour, on horizontal rails which lie on a curve of 250 metres radius. Find the horizontal stress between the truck and a mass of 50 kilogrammes lying in it.

3. Supposing the earth to be a sphere of radius 3969 miles, and to rotate once in 24 hours; by how much will the apparent acceleration of gravity at the equator be less than it would be if there were no rotation? [$\pi = \frac{22}{7}$.]

4. In worked example **D**, find the tension of the string.

5. Why is it necessary to lean inwards when going quickly round a corner on a tricycle?

6. At a place where the acceleration of gravity is 32·2 sfas, a particle is attached to one end of a string, 5 feet long, the other end of the string being attached to a point in a smooth table; and the particle is set to describe a circle on the table with a velocity of 23 fas. What is the greatest possible mass that the particle can have, if the string is just strong enough to carry (at rest) a mass of 23 lbs.?

7. Taking the earth a sphere of radius 6400 kilometres, and the moon as revolving in $27\frac{1}{2}$ days in a circle round the earth, at a distance (centre to centre) of 384,000 kilometres; and assuming that the earth attracts external bodies with a force varying inversely as the square of the distance from its centre; what does this give for the acceleration of gravity at the surface of the earth? [$\pi = \frac{22}{7}$.]

8. If the length of a Conical Pendulum be 4 ft. 1 inch at a place where the acceleration of gravity is 32·0166 sfas; find the inclination of the string to the vertical when the bob is making 63 revolutions in 100 seconds. [$\pi = \frac{22}{7}$.]

---

9. Why does a string, which is strong enough to carry a mass when it is stationary, break sometimes when the mass swings as a pendulum?

10. Two masses of 2 lbs. and $4\frac{1}{2}$ lbs. are attached to the ends of a massless string, 8 feet long, which passes through a small hole in a smooth table. In what proportion must the string be divided by the hole, that when the mass of 2 lbs. is set revolving

with a velocity of 17 fas in a circle round the hole as centre, there may be equilibrium? [$g=32\frac{1}{5}$.]

11. In question 3, if the known acceleration of gravity at the equator were 31·944 sfas, in what time would the earth have to rotate that matter at the equator might have no weight?

12. In question 8, if the mass of the bob be 1 lb., find in lbs. weight the tension of the string.

---

13. If the earth were once fluid, show that its present shape is the natural outcome of its rotation.

14. At a place where the acceleration of gravity is 32·12 sfas, a mass of 73 lbs. is swinging by a massless string 11 feet in length: find the strain on the string, in addition to the weight, when the mass is at its lowest point, and is moving at the rate of 15 miles an hour.

15. Supposing the earth to be a sphere of radius 3969 miles, and to rotate once in 24 hours; by how much is the acceleration toward the centre, in latitude 30°, less than it would be if there were no rotation? [$\pi=3\frac{1}{7}$.]

16. Show that a conical pendulum which revolves at an inclination to the vertical of $\cos^{-1}\frac{1}{4}$, revolves in the same time as a simple pendulum of the same length would oscillate.

SECTION XV.

PROOFS OF THE LAWS.

303. We have now before us a fair number of tests, direct and indirect, of the laws of motion.

304. It is evident from what has preceded that these laws, though they may be in a sense called axioms, are not obvious at first sight.

There are axioms which are, at once, entirely obvious; as most of those in Euclid's 1st book.

But the truth of the 12th Axiom cannot be fully appreciated until the learner has mastered the truth contained in the 28th Proposition.

We have to go even further before we can be sure of the truth of the laws of motion.

305. One of the real difficulties is that in a certain rough way the laws *seem* entirely obvious.

The 1st law, *e.g.*, is often put thus: "Change of motion cannot take place without external force;" and if a mere movement is meant this is fairly obvious, though not entirely so; nor is it entirely true. See Section v., Examples **E**, 5, 14, etc.

Take the case of a system of two colliding balls. We know, as a fact (see Section IV., Example 26), that there is a change of one kind of motion, viz., of kinetic energy; and therefore, to avoid ambiguity, the law should read,—"change of momentum cannot take place without external force."

But directly this word, momentum, is introduced, and a proper definiteness is given to the law, the seemingly obvious truism of the law vanishes.

306. The laws then are not obvious; and the learner who thinks that he can see the truth of them at once is but deceiving himself.

307. There are, indeed, many facts which a careful reader will note as tending, without explanation, to show that the laws are *not* true.

Thus the correct method of stating the 1st law is that the centre of mass of a body cannot move itself; and yet a boy can swing himself to a great height.

It is in answering such objections as these (see Section V., Examples F, 6, 15, etc.), that the laws have triumphed. No difficulty has been raised which has not been successfully answered.

308. THE 1ST LAW OF MOTION.—The 1st law may be looked upon as merely clearing the ground for the 2nd.

The 2nd being true, the 1st must also be true. For the momentum produced in a given time being proportional to the force, it follows that if there is no force, there is no

momentum being produced; and the mass simply retains the momentum it already has.

309. The truth of the 1st law is therefore involved in that of the 2nd.

Nevertheless it is convenient to endeavour to test separately the truth of the 1st law.

Many instances of such testing may be found in the examples in Section V.

310. A notable test of the truth of the 1st law is given by the Theory of Impact.

In the impacts of bodies, the momentum gained by one body is proved by Newton's experiments to be equal to that lost by the other; *i.e.* the total momentum of the system remains constant.

Now this is precisely the 1st law of motion. When there is no external force the total momentum of the system remains constant.

The force of this proof of the 1st law is in this, that it is the total *momentum* (Newton found) that remains constant; not the total *kinetic energy*, or any other function of the mass and velocity. In fact the total kinetic energy is always reduced by collision; some being "lost" in heat.

311. THE 2ND LAW OF MOTION. Perhaps the best simple test of the 2nd law is the parallelogram of forces.

It is clearly shown in Section X. that the parallelogram of forces does not maintain unless the 2nd law is true in both its parts.

But if there is one thing which enters into nearly all investigations as to the results of the action of forces, it is the composition and resolution of forces.

No result has ever been found fault with on account of its dependence on the parallelogram of forces.

This is really in itself a complete test (I may say proof) of the 2nd law.

312. Many simple tests of the truth of the 2nd law will be found in the examples in Section VI.; and the experiment on the parallelogram of blows in Section X. is also a good test.

313. When two bodies P and Q hang over a smooth massless pulley, the heavier P descends, and the velocity generated in a given time is as $P-Q$ directly and as $P+Q$ inversely.

This is proved by Atwood's machine; and is a test of the truth of the 2nd law. For the difference of P and Q is clearly proportional to the force acting; while the sum of P and Q is proportional to the whole mass acted on;

$$\therefore \text{ velocity} \propto \text{force directly};$$
$$\text{and} \propto \text{mass inversely};$$

which is the 2nd law.

314. Example **F** in Section VI., tested by such experiments as those contained in examples 6, 13, and 34 in the same section, is additional testimony to the truth of this law.

315. THE 3RD LAW OF MOTION. The 3rd law is far more complicated and difficult than is the 2nd.

So much so, that it is usual to abbreviate the law into

"*Action and Reaction are equal and opposite*;" but as *Action* and *Reaction* are left vague and undefined, this is imperfect and unsatisfactory.

The illustrations usually given, such as that *a finger pressing a table is pressed by the table with the same force as it presses*, etc., seem to imply that by *Action* and *Reaction* is understood simply *Force* or the *Momentum* arising from force.

And indeed some writers deduce the equation

$$mV + m'V' = mv + m'v',$$

(for two colliding bodies) from this law; but this surely is but stating the 1st law in another form.

316. Newton, it is true, gives the law something in this shape; but then he proceeds to define *Action* and *Reaction* as in Example **E** in Section VII.

Those who refuse this are obliged to introduce, later on, what is called D'Alembert's Principle; which is only the 3rd law of motion as given at the head of Section VII.

317. Now by D'Alembert's Principle an immense number of problems are easily solved which previously almost defied the skill of mathematicians.

Before the publication of this principle the solution of problems required the exercise of ingenuity and skill to detect a suitable principle applicable to the case. Such problems were a sort of trial of strength among mathematicians; but the *Traité de Dynamique*, published by D'Alembert in 1743, put an end to these challenges.

The true solution of such problems by D'Alembert's principle is a complete test of the truth of that principle; and therefore, of course, of the truth of the 3rd law.

318. Joule's investigations, which show that the work done against Friction (which in old times was thought to be lost) has a perfectly definite equivalent in the heat produced, have finally settled the complete accuracy of this law.

319. Galileo's experiment, which showed that *on an inclined plane the velocity acquired by falling down the plane is the same as that acquired by falling freely down the vertical height of the plane*, is a simple test of the 3rd law. The work done in either case is clearly the same; and therefore the energy exhibited should be the same.

320. The pendulum, again, affords a test of the 3rd law. The results obtained in Section XI. have all been verified.

The result that *the small oscillations of pendulums are performed in times which are as the square roots of the lengths of the pendulums; but which are independent of the angle through which the pendulum swings*, is the foundation on which all pendulum clocks are made.

Galileo, it is said, tested the latter part of this by noticing the swaying of a bronze lamp (still shown) in the Cathedral at Pisa, and measuring the times by his pulse.

It can be easily tested; see Section XI., Examples **C**, 3, 12, etc.

321. The projectile results obtained in Section XIII. are so vitiated in practice by the resistance of the air as to afford no test.

322. But the planets move in a medium which offers little or no resistance to their motion; and the marvellous work done on the basis of the laws by Astronomers (of which a simple example is given in Section VI.) is a convincing test of the truth of the Laws of Motion.

ANSWERS TO THE EXAMPLES.

I.

1. $30{\cdot}479\dot{3}$.	**2.** 15400.	**3.** 88.
4. 100.	**5.** $111\frac{1}{9}$.	**6.** ·032808.
7. $312\frac{1}{2}$.	**8.** 10″.	**9.** 99″.
10. ·12192.	**11.** ·393696.	**12.** $2857\frac{1}{7}$.
13. 48″.	**14.** 19·8.	**15.** $62\frac{1}{2}$.

II.

1. ·0353.	**2.** $\frac{1}{2}$.	**3.** 28·372625.
4. $1 : 62\frac{1}{2}$.	**5.** $\frac{1}{5}$ foot.	**6.** $\cdot 7958\dot{3}$.
7. 4543·8.	**8.** $4252\frac{1}{2}$.	**9.** 11.
10. $1\frac{1}{4}$.	**11.** 54.	**12.** 3.
13. 1·806448 pints.	**14.** 20.	**15.** $12(5280)^3$lbs.; 1 mile.
16. 125 : 126.	**17.** 40 cub. centimetres; $7\frac{1}{2}$.	
18. ·525 oz.; ·104 oz.		

III.

1. 1760.	**2.** 976.	**3.** 242 ft.; $2''\frac{3}{4}$.
4. 1·21.	**5.** 144·54.	**6.** $29129{\cdot}\dot{6}2\dot{9}$.
7. 38400.	**8.** 4″·54.	**9.** 1125.
10. 4410.	**11.** 187·902	**12.** 11″.
13. $178\frac{1}{2}$ feet; 107·1 fas.		**16.** 30.
17. $\frac{240}{11}$.	**18.** $\frac{FT_2}{V}$ seconds; $\frac{2LFT_2^{\,2}}{V^2T_1^{\,2}}$ feet.	

19. 910 cas; $422\frac{1}{2}$ cms. 20. $4'' \frac{19}{38}$.
21. $1521\frac{2}{3}$; $46410\frac{5}{6}$. 22. 32·12. 23. 1 minute; 1 metre.
24. $\frac{1}{2}\frac{l^2\tau^2}{\lambda t^2}$. 25. 1 mile; 10″.

IV.

1. 13825·37 . . . 2. 4; 1. 3. 81 scas.
4. 5″. 5. 1000 cas; 3″. 6. 16.
7. 800 cas. 8. $\frac{1}{7}$; $\frac{2}{7}$; $\frac{4}{7}$. 9. $\cdot 00007\dot{3}$.
10. 15 lbs. 11. Equal; 3 : 1. 12. 5 kilogrammes.
13. 29 fas; 36 fas. 15. $897\frac{3}{4}$ scas. 16. 25.
17. 2. 18. 10″. 19. 4 oz.; 12 oz.
20. 421393. 21. $\frac{1}{10}$ sfas. 22. 18×10^6; 48×10^9; 3×10^5.
24. 258″·72; 15. 222 25. 170 cas; 270 cas. 27. 9 scas.
28. 5″; 275 ft.; 2 lbs.; 110 fasp˙; 6050 faspen.
29. 10^{11} scasgram. 30. 400″; 43·2.

V.

1. The body retains its motion, while the feet are stopped. 2. Inertia in each case. 5. $\frac{3}{8}$ inch. 6. He has a point d'appui at the beam above; so he is able to judiciously lower and raise his centre of mass. 7. $2\frac{1}{4}$ cas. 8. The book has two velocities, that of the throw and that of the train; but we who also have the train velocity only perceive the one. 9. 45°. 10. It is twisted to give more permanence to the motion. 11. The friction is insufficient to overcome the inertia of the door. 12. 5 ft. from 0. 14. He is giving a quick backward motion to the centre of mass. 15. The friction at the point d'appui gives time for throwing a foot forward, on ground though not on ice. 17. The throw cannot interfere with the action of gravity. 19. Friction of cloth and resistance of air. 20. The friction of the card is not sufficient to overcome the inertia of the shilling. 21. 4 feet. 23. 4 inches. 24. The balancing momentum is communicated to the earth, but its enormous mass makes the change of velocity invisible.

25. 44. 26. The only force is the pressure on the surface; this being along the normal cannot affect the motion. 27. $V\sqrt{\frac{2h}{g}}$ feet. 28. The outer; to balance the tendency to rectilinear motion. 29. Backward; owing to the inertia of the water. 30. 1 foot. 31. Distance from centre : radius : : $1+\sqrt{3} : 3+\sqrt{3}+\sqrt{2}$. 32. 1″. 33. The friction and ground pressure enable him to spring at an angle inclined to the vertical. 34. 5 fas; 25 fas. 35. The vertical throw cannot interfere with the ball's horizontal velocity. 36. $\frac{1000}{3}$; 3270; 10000. 37. See Answer 10. 38. The handle is suddenly stopped, and the inertia of the head drives it home. 39. 1 foot. 40. 2 : 1. 41. 4 : 3; the lighter; $7\frac{1}{2}$ feet. 42. Because of its points d'appui. 43. 4 : 1. 44. Her momentum forward is not interfered with by her upward jump. 45. 258 ft.; $161\frac{1}{4}$ fas.

VI.

1. 150000 : 11. 2. Of *any* two such triangles that which has the greater base is the greater; which satisfies Euclid v., Def. 5. 3. See worked question C. 4. 21 : 29. 5. 3. 6. See worked question F, the ordinary air-resistance on the paper being here eliminated. 7. 31206 days. 8. 36 kilogrammes. 9. See answer to question 2. 10. The force being the same, the greater the mass the less the velocity. 11. The greater the height the greater the momentum, and therefore the greater the blow. 12. One million. 13. $gt-\frac{R}{m}>gt-\frac{R}{m'}$. 14. 1 hr. 31 min. 24 sec. 15. 2 lbs. 16. See worked question B. 17. With the free nail there is less mass to move. 18. Cf. worked question D. 19. 20″. 20. It is only true because two constant forces acting in constant directions for the same time on the same mass, are proportional to the lengths described. 21. 900 : 1521. 22. 50″. 23. No; for then Euclid v., Def. 5 would make ft $\propto \frac{1}{2}mv^2$. 24. The mass is now proportionately less; cf. worked question D. 25. At first the momentum is being rapidly *increased;* while afterwards only sufficient production of momentum is required to overcome friction.

26. 51392 poundals. 27. 32·044 fas. 28. 8 hr. 11 min. 42 sec. 29. 5′ 8″. 30. $fx \propto \frac{1}{2}mv^2$; by Euclid v., Def. 5. 31. The smaller the mass of the chisel, the greater the velocity his blow will produce in the chisel. 32. The forces are proportional to the *total* masses. 33. 90 metres. 34. Apart from the resistance of the air, all bodies fall to the ground in the same time. 35. 4096″.

VII.

1. 1606 feet. 2. 654000. 3. $m(g-a)$ poundals. 4. Each is $\sqrt{2g\left(x+\frac{V^2}{2g}\right)}$ cas. 5. There must be some heat developed by air-friction and the compression of the marble. 6. $\frac{1}{2}mv^2 = mg \sin a \times l = mgh$. 7. 9800000. 8. 98 scas; 1400 cas. 9. 36 poundals. 10. $6\frac{1}{4}$ feet. 11. That they are in equilibrium among themselves. 12. It has its equivalent mainly in the potential energy laid up ready to be converted into the kinetic energy of the arrow. 13. 219 lbs. 14. See answer to question 6. 15. m lbs. 16. 155 cas. 17. Friction, etc., converted into heat. 18. The third law becomes zero = K.E. + work done against gravity = K. E. + P. E. 19. 1400 cas. 20. 14 fas. 21. Zero. 22. 962 cas. 23. See answer to question 5. 24. When the agents acting on a mass are doing no work, the sum of the Potential and Kinetic energies is constant. 25. $\frac{1}{2} \times 10^9$ dynes; ·002″. 26. $pt : px$. 27. The apparent weight will be doubled. 28. 5880000. 29. Against the winners, as they have to give it kinetic energy. 30. See question 18. 31. 46·8952. 32. $654000 \pm \frac{109000}{667}$. 33. The work done in passing from one point to another $= fa - fa = 0$; $\therefore$ the difference of the kinetic energies is also zero. 34. $a = \frac{\text{K. E. gained}}{\text{incr. of vector}}$. 35. Now, owing to tide-friction, there is continual loss of energy; which loss would then vanish. 36. $\frac{1}{2}m(V-gt)^2$; $mg(Vt - \frac{1}{2}gt^2)$.

VIII.

1. $5'\ 8''$. **2.** 2 lbs. **3.** 13847.

4. 5 : 6. **5.** 981 ; 98100000. **6.** $5''$; $200\frac{3}{4}$ feet.

7. 80. **8.** 10 lbs. $\frac{1}{2}$ oz. **9.** $1''$.

10. $44\frac{1}{2}$. **11.** 50000. **12.** 13395600 ; $418612\frac{1}{2}$.

13. $1250''$; 50187500 feet. **14.** 2240.

15. $\frac{31}{175}$ sfas. **16.** $12\frac{1}{2}$. **17.** 1400000.

18. 4 : 7. **19.** 70. **20.** 1 yard.

21. 40 fas. **22.** $2'\ 55''$. **23.** 13827.

24. 161. **25.** 3×10^9.

26. $\cdot 0000000763\dot{8}$; $\cdot 000002\dot{4}$. **27.** 1 inch ; $\frac{2}{3}$ lbs.

28. 1200 cas. **29.** 4·6 sfas. **30.** $\frac{10}{313}$ second.

31. 1 : 2. **32.** 4 oz. ; 12 oz. **33.** 1.

34. 1210 hours. **35.** 7,447,093,500. **36.** $\frac{2}{5}$ foot ; 2 lbs.

37. $\cdot 3195''$. **38.** 1 : 4.

39. 250 grms. ; 500 grms. ; 250 grms. **40.** 25 billion ergs.

41. 1296·05 ; 25·3. **42.** ·0375 sfas.

IX.

1. 0. **2.** $10''$.

3. $\frac{u^2}{2g}$ cms. ; $\frac{u}{g}$ seconds ; $\frac{1}{2}mu^2$ ergs ; $\frac{u}{g}$ seconds ; u cas.

4. $3''$. **5.** 15528 feet. **6.** 2000000 dynes.

7. 3824 grammes. **8.** $2''$. **9.** 6.

10. $10''$. **11.** $402\frac{1}{2}$ ft. ; $5''$. **12.** 16.

13. 2000 cas. **14.** $1''\frac{1}{2}$; $1102\frac{1}{2}$ cms. **15.** 50 : 73.

16. 560000 dynes. **17.** $\cdot 001''$. **18.** $18\frac{9}{16}$.

19. $20''$. **20.** $7''\frac{1}{2}$; 1509·375 ft. **21.** 40 fas.

22. $28\frac{3}{4}$ feet. **23.** $10000\,(5 - \sqrt{24})$. **24.** 295680.

25. $\left(\frac{m - m'}{m + m'}\right)^2 g$ sfas. **26.** $2''\frac{3}{4}$; $\frac{\sqrt{473}}{4}$ seconds.

27. 7 : 2. **28.** $100''$. **29.** $2\frac{7}{9}$.

30. 5·625. 31. 981 dynes; 1000 cas. 32. 2″. 33. 196 ft. 34. 2 minutes. 35. 4032 fasp. 36. 1600 fas; ·00125″.

X.

1. 2·6 fas. 2. $\frac{160}{3}$ fas. 3. 3 fas; at right angles. 4. 19. 5. DC. 6. 3 fasp. 7. The relative velocity is the same in either case; and there is nothing to guide him until jerking begins. 8. Proof as on page 145. 9. 55 fas. 10. 5 miles an hour; due North. 11. 70 miles an hour. 12. 7 sfas. 13. $AB\sqrt{4+2\sqrt{2}}$; parallel to AE. 14. 250000 casgram towards centre of hexagon. 15. From the last words in the law. 16. Proof as in the text. 17. 60°. 18. At right angles. 19. 28 miles an hour. 20. One half. 21. $a:b:c$. 22. 60°; 60 fas. 24. $\frac{1}{2}mv^2+\frac{1}{2}mv'^2$ is not equal to $\frac{1}{2}m(v+v')^2$. 25. 37 fas. 26. 60°. 27. 45. 28. 3. 29. Proportional to, and at (h_2-h_1) feet from, BC. 30. 78 fas; 117 feet. 31. Treat as worked example C. 32. Proof as in the text. 33. 72″. 34. $\frac{V^2}{a}$ sfas. 35. $7\frac{1}{2}$ miles an hour. 36. $2''\frac{1}{2}$. 37. $\theta>120°$. 38. $2\sin\frac{\pi}{n}$ fasp. 39. See answer to question 7. 40. Demonstration similar to that in text.

XI.

1. $2\sqrt{lg}\times\sin\frac{a}{2}$ cas. 2. 32·19. 3. 7. 4. 576 cms. 5. Keep m and m' constant; and show (by experiment) that velocity below ring $\propto\frac{1}{t}$. 6. $8\frac{13}{14}$ grammes. 7. 980 scas. 8. See worked Example H; a feet. 9. 32·11 sfas. 11. 180. 12. Cf. qu. 10; they fall in the same time, and come to rest. 13. 1 metre. 14. $\frac{(2m+m')a}{m'rt}$. 15. 1·1 metre. 16. 32·2 sfas. 17. Draw a circle touching the

two, and a chord in it parallel to the given chord. 18. $\frac{1}{2}$.
19. $v^2 = 2g \cdot AM\left(1 - \frac{AN}{AM}\right)$. 20. $\frac{22''}{49}$. 22. $1''\cdot 2006$. 23. Keep m' and t constant. 24. 981 scas. 25. Equal to that of gravity.
26. $\frac{AB}{AP}\sqrt{\frac{a}{g}}$ seconds. 27. 30°. 28. $\frac{R+h}{R}$ seconds. 29. 1 mile.
30. 81 cms.; try practically with 39 and 31 inches. 31. ·1199 inch.
32. $\frac{(2m+m')a^2}{2m'lt^2}$. 33. 11 feet. 34. 119 tons 19 cwt. 35. Let AB meet the horizontal through P in C, and take CQ (on AB) $= CP$; PQ is the required line. 36. 1461·36; 499 tons 19 cwt. 38. 6350·4 kilometres. 39. They fall in the same time; and move together in the direction of motion of the greater mass. 40. $6\frac{1}{5}\frac{2}{0}\frac{2}{5}$. 41. Take $2m + m'$ constant, but vary m'; and keep constant the time above the ring. 42. 4″. 43. 320.

XII.

1. Take the velocities equal at the moment of greatest compression.
2. 35 and 31 fas in opposite directions. 3. 30°; 45°. 4. 7 fas.
7. $\frac{1}{\sqrt{3}}$; 45°. 8. $\frac{m + m' \sin^2 a}{\cos^2 a}$. 9. $\frac{m'}{m}$. 10. 6, 6, 6 fasp; 3, 6, 18 faspen. 11. $\frac{1}{\sqrt{14}}$. 12. 19 and 11 fas. 14. $m = (3 \pm 2\sqrt{2})m'$; the double sign because the kinetic energies may be in the same or opposite directions. 16. $-\frac{v}{5}$, $\frac{2v\sqrt{3}}{5}$ and $\frac{2v\sqrt{3}}{5}$ fas.
17. $(1-e^2)\frac{mm'}{2(m+m')}(v'-v)^2$, no loss when $e=1$. 18. $m : (m+m')$.
20. 40 fasp; 200 faspen; no change; 10 fas. 21. $\sqrt{mm'}$ lbs.
22. 10 : 6 : 3 : 1. 23. With equal velocity, into the pocket at B.
24. Along and at right angles to the line joining their centres.

XIII.

1. $v^2 = (V\cos a)^2 + (V\sin a - gt)^2$. 2. 32·12 sfas. 3. $2a - \frac{\pi}{2}$.

4. 1 : 2. 5. $\frac{V^2}{gh} \pm \sqrt{\frac{V^4}{g^2h^2} - \frac{2V^2k}{gh^2} - 1}$; $gk + g\sqrt{h^2+k^2}$.

7. $v^2 = V^2 - 2gY$. 8. $5208\frac{1}{3}$ ft. ; $5''\frac{5}{24}$; $108\frac{73}{144}$ feet.

11. $2\sqrt{\frac{2H}{g}}$ seconds. 12. 1 : $\sqrt{3}$; 1 : $3(2-\sqrt{3})$. 13. The line is at a height $\frac{V^2}{2g}$ feet. 14. 72 feet. 16. 45° ; 900 feet ; 1200 yards ; in horizontal plane. 17. $\frac{h}{h\sin a - k\cos a}$. 18. 315 cas.

19. Vt feet (see worked Example C). 20. 40 metres ; $1400\sqrt{2}$ cas.

XIV.

1. The normal mass-acceleration on any particle soon overcomes the constraining pressure of the surrounding fluid. 2. 3,125,000 dynes. 3. ·110916 sfas. 4. $\frac{4\pi^2 ml}{\tau^2}$ poundals. 5. The centre of mass (which is very high) may otherwise be taken too far out. 6. 7 lbs. 7. $967\frac{659}{1323}$ scas. 8. 60°. 9. The acceleration due to the circular motion is added to the acceleration of gravity. 10. Bisected. 11. One-seventeenth of a day. 12. 2. 13. The equatorial parts would bulge until the normal acceleration balanced the acceleration of gravity. 14. 100 lbs. weight. 15. ·0831875 sfas.

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